

SHEAFIFICATION FUNCTORS AND TANNAKA'S RECONSTRUCTION

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ABSTRACT. We introduce “sheafification” functors from categories of (lax monoidal) linear functors to categories of quasi-coherent sheaves (of algebras) of stacks. They generalize the homogeneous sheafification of graded modules for projective schemes and have applications in the theory of non-abelian Galois covers and of Cox rings and homogeneous sheafification functors. Moreover, using this theory, we prove a non-neutral form of Tannaka’s reconstruction, extending the classical correspondence between torsors and strong monoidal functors.

INTRODUCTION

If G is an affine group scheme over a field k , classical Tannaka’s reconstruction problem consists in reconstructing the group G from $\text{Rep}^G k$, its category of finite representations: if $F: \text{Rep}^G k \rightarrow \text{Vect } k$ is the forgetful functor then G is canonically isomorphic to the sheaf of automorphisms of F (opportunistically defined, see [DM82, Proposition 2.8]). More generally one can recover the stack BG of G -torsors by looking at fiber functors. Given a ring A denote by $\text{Loc } A$ the category of locally free sheaves of finite rank over A , that is finitely generated projective A -modules. If SMon_k^G is the stack over k whose fiber over a k -algebra A is the category of k -linear, strong monoidal and exact functors $\Gamma: \text{Rep}^G k \rightarrow \text{Loc } A$ then the functor

$$(\star) \quad \text{BG} \rightarrow \text{SMon}_k^G, (\text{Spec } A \xrightarrow{s} \text{BG}) \mapsto (s_{|\text{Rep}^G k}^*: \text{Rep}^G k \rightarrow \text{Loc } A)$$

is an equivalence of categories (see [DM82, Theorem 3.2]). Here $\text{Rep}^G k$ is thought of as the category of locally free sheaves of finite rank on BG .

The above functor can be defined in a way more general context. Let us introduce some notations and definitions. We fix a base commutative ring R and a category fibered in groupoids \mathcal{X} over R . We say that \mathcal{X} is *pseudo-algebraic* (resp. *quasi-compact*) if there exists a scheme (resp. affine scheme) X and a map $X \rightarrow \mathcal{X}$ representable by fpqc covering of algebraic spaces. We denote by $\text{QCoh } \mathcal{X}$ and $\text{Loc } \mathcal{X}$ the categories of quasi-coherent sheaves and locally free sheaves of finite rank respectively (see Section 1). Given a full subcategory \mathcal{C} of $\text{QCoh } \mathcal{X}$ we say that \mathcal{C} generates $\text{QCoh } \mathcal{X}$ if all quasi-coherent sheaves on \mathcal{X} are quotient of a (possibly infinite) direct sum of sheaves in \mathcal{C} . We say that \mathcal{X} satisfies (or has) the *resolution property* if $\text{Loc } \mathcal{X}$ generates $\text{QCoh } \mathcal{X}$.

Given a monoidal and additive full subcategory \mathcal{C} of $\text{Loc } \mathcal{X}$ and a category fibered in groupoids \mathcal{Y} over R define $\text{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y})$ as the category of R -linear and strong monoidal functors $\Gamma: \mathcal{C} \rightarrow \text{Loc } \mathcal{Y}$ which are exact on right exact sequences in \mathcal{C} (in the ambient abelian category $\text{QCoh } \mathcal{X}$). Denote also by $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ the stack over R whose fiber over an R -algebra A is $\text{Fib}_{\mathcal{X}, \mathcal{C}}(\text{Spec } A)$ and by $\mathcal{P}_{\mathcal{C}}$ the functor

$$\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}, (\text{Spec } A \xrightarrow{s} \mathcal{X}) \mapsto (s_{|\mathcal{C}}^*: \mathcal{C} \rightarrow \text{Loc } A)$$

The functor (\star) is obtained by taking $R = k$, $\mathcal{X} = \text{BG}$ and $\mathcal{C} = \text{Loc } \mathcal{X}$. We prove the following non-neutral form of Tannaka’s reconstruction.

Theorem (5.3, 5.4). *Let \mathcal{X} be a quasi-compact stack over R for the fpqc topology with quasi-affine diagonal and $\mathcal{C} \subseteq \text{Loc } \mathcal{X}$ be a full, additive and monoidal subcategory with duals generating $\text{QCoh } \mathcal{X}$.*

If $\Gamma: \mathcal{C} \rightarrow \text{Mod } A$, where A is an R -algebra, is an R -linear, contravariant and strong monoidal functor such that Γ , as well as $\Gamma \otimes_A k$ for all geometric points $\text{Spec } k \rightarrow \text{Spec } A$, is left exact on right exact sequences in \mathcal{C} then there exists $\text{Spec } A \xrightarrow{s} \mathcal{X}$ such that $\Gamma \simeq (s^)^\vee$.*

The functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$ is an equivalence of stacks and, if \mathcal{Y} is a category fibered in groupoids, the functor

$$\text{Hom}(\mathcal{Y}, \mathcal{X}) \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y}), (\mathcal{Y} \xrightarrow{f} \mathcal{X}) \mapsto f|_{\mathcal{C}}^*: \mathcal{C} \rightarrow \text{Loc}(\mathcal{Y})$$

is an equivalence of categories.

Notice that the two conclusions in the last statement are equivalent. In the case $\mathcal{C} = \text{Loc } \mathcal{X}$, the functor $\mathcal{P}_{\text{Loc}(\mathcal{X})}$ has already been proved to be an equivalence in the neutral case, that is $\mathcal{X} = \text{B}_R G$, where G is a flat and affine group scheme over R (see [Bro13, Theorem 1.2], where R is a Dedekind domain, and [Sch13, Theorem 1.3.2] for general rings R), for particular quotient stacks over a field (see [Sav06] and 5.12) and for quasi-compact and quasi-separated schemes (see [BC12, Proposition 1.8]). We also show an almost converse of Theorem above:

Theorem. [5.7] *Let \mathcal{X} be a quasi-compact category fibered in groupoids over R admitting a surjective (on equivalence classes of geometric points) map $X \rightarrow \mathcal{X}$ from a scheme whose connected components are open (e.g. a connected or Noetherian algebraic stack) and let $\mathcal{C} \subseteq \text{Loc } \mathcal{X}$ be a full monoidal subcategory with duals such that $\text{Sym}^n \mathcal{E} \in \mathcal{C}$ for $n \in \mathbb{N}$ and $\mathcal{E} \in \mathcal{C}$ if \mathcal{E} has local rank not invertible in R (e.g. $\mathcal{C} = \text{Loc}(\mathcal{X})$ or \mathcal{C} consists of invertible sheaves). Then $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is a quasi-compact stack in groupoids for the fpqc topology over R with affine diagonal and with a collection of tautological locally free sheaves $\{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}}$ generating $\text{QCoh}(\text{Fib}_{\mathcal{X}, \mathcal{C}})$ and such that $\mathcal{P}_{\mathcal{C}}^* \mathcal{G}_{\mathcal{E}} \simeq \mathcal{E}$.*

For instance it follows that $\mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \text{Loc}(\mathcal{X})}$ is universal among the maps $w: \mathcal{X} \rightarrow \mathcal{Y}$ where \mathcal{Y} is a quasi-compact stack over R with quasi-affine diagonal and the resolution property.

There are also variants of theory above where $\text{Loc}(-)$ is replaced by $\text{Coh}(-)$ or $\text{QCoh}(-)$ or the derived category $D(-)$ (see [Lur04, Sch12, BC12, Bra14, Bha14]). Although not explicitly stated elsewhere, for stacks with the resolution property and with affine diagonal (which is automatic in the algebraic case, see [Tot04]) those results and the fact that $\mathcal{P}_{\text{Loc}(\mathcal{X})}$ is an equivalence can be proved to be equivalent: one can pass from quasi-coherent sheaves to locally free sheaves via dualizable objects and, for the converse, extend functors from $\text{Loc}(-)$ to $\text{QCoh}(-)$ following the proof of [Bha14, Corollary 3.2]. We complete this picture by showing that in general the resolution property implies the affineness of the diagonal (see 5.13). One of the ingredients in the proof is the classification of quasi-compact stacks whose quasi-coherent sheaves are generated by global sections, called pseudo-affine. In the algebraic case those coincide with quasi-affine schemes (see [Gro13, Proposition 3.1]), while, in general, we prove they are (arbitrary) intersection of quasi-compact open subschemes (thought of as sheaves) of affine schemes (see 5.6).

The proof of Tannaka's reconstruction we present does not reduce to the case of quasi-coherent sheaves as explained above but it follows a different path. It is obtained by developing a theory of sheafification functors which I think is interesting on its own and it is the heart of the paper. In what follows A will denote an R -algebra, \mathcal{X} a category fibered in groupoids over R and \mathcal{C} a full subcategory of $\text{QCoh } \mathcal{X}$. The idea is simple: if $s: \text{Spec } A \rightarrow \mathcal{X}$ is a map (say quasi-affine so that s_* preserves quasi-coherency) then we have natural isomorphisms of A -modules

$$s^* \mathcal{E} \simeq \text{Hom}_A(s^* \mathcal{E}^\vee, A) \simeq \text{Hom}_{\mathcal{X}}(\mathcal{E}^\vee, s_* \mathcal{O}_{\text{Spec } A})$$

By passing from covariant functors to *contravariant* ones we can always define

$$\Omega^{\mathcal{F}}: \mathcal{C} \longrightarrow \text{Mod } A, \quad \Omega_{\mathcal{E}}^{\mathcal{F}} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) \text{ for } \mathcal{F} \in \text{QCoh}(\mathcal{X} \times A)$$

which are R -linear contravariant functors. If $\text{L}_R(\mathcal{C}, A)$ is the category of contravariant R -linear functors $\mathcal{C} \longrightarrow \text{Mod } A$ we obtain a functor

$$\Omega^*: \text{QCoh}(\mathcal{X} \times A) \longrightarrow \text{L}_R(\mathcal{C}, A)$$

Quite surprisingly (since we started from group schemes as motivation) we recover also this well known situation (see 2.2 for details). If X is a quasi-projective and quasi-compact scheme over R with very ample invertible sheaf $\mathcal{O}_X(1)$ consider $\mathcal{C}_X = \{\mathcal{O}_X(n)\}_{n \in \mathbb{Z}}$, set S_X for the homogeneous coordinate ring of $(X, \mathcal{O}_X(1))$ and use $\text{GMod}(-)$ to denote the category of graded modules. We have that $\text{L}_R(\mathcal{C}_X, A)$ is equivalent to $\text{GMod}(S_X \otimes_R A)$ and Ω^* corresponds to

$$\Gamma_*: \text{QCoh}(X \times A) \longrightarrow \text{GMod}(S_X \otimes_R A), \quad \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n))$$

It is a classical result, at least over a field, that the above functor is fully faithful. Thus in general we can ask under what conditions the functor $\Omega^*: \text{QCoh}(\mathcal{X} \times A) \longrightarrow \text{L}_R(\mathcal{C}, A)$ is fully faithful as well. Another property that the functor Γ_* has is the existence of a left adjoint, namely the homogeneous sheafification functor $\widetilde{\cdot}: \text{GMod}(S_X \otimes_R A) \longrightarrow \text{QCoh}(X \times A)$. It turns out that also $\Omega^*: \text{QCoh}(\mathcal{X} \times A) \longrightarrow \text{L}_R(\mathcal{C}, A)$ has a left adjoint $\mathcal{F}_{*, \mathcal{C}}: \text{L}_R(\mathcal{C}, A) \longrightarrow \text{QCoh}(\mathcal{X} \times A)$ when \mathcal{C} is essentially small: by analogy we call this functor a *sheafification* functor.

In the context of Tannaka's reconstruction we also have that sheaves of algebras correspond to (lax) monoidal functors $\text{Loc}^G R \longrightarrow \text{Mod } A$. This is true in general. If \mathcal{C} is a monoidal subcategory of $\text{QCoh } \mathcal{X}$, $\text{ML}_R(\mathcal{C}, A)$ denotes the category of R -linear, contravariant and monoidal functors $\mathcal{C} \longrightarrow \text{Mod } A$ and $\text{QAlg}(\mathcal{X} \times A)$ the category of quasi-coherent sheaves of algebras on \mathcal{X} then Ω^* extends to a functor $\Omega^*: \text{QAlg}(\mathcal{X} \times A) \longrightarrow \text{ML}_R(\mathcal{C}, A)$ and, if \mathcal{C} is essentially small, $\mathcal{F}_{*, \mathcal{C}}$ extends to $\mathcal{A}_{*, \mathcal{C}}: \text{ML}_R(\mathcal{C}, A) \longrightarrow \text{QAlg}(\mathcal{X} \times A)$, which is still a left adjoint of Ω^* .

Coming back to the fully faithfulness of Ω^* we prove the following.

Theorem (3.12, 3.18). *Let \mathcal{X} be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C} \subseteq \text{QCoh } \mathcal{X}$ be a full subcategory generating $\text{QCoh } \mathcal{X}$. Then the functor $\Omega^*: \text{QCoh}(\mathcal{X} \times A) \longrightarrow \text{L}_R(\mathcal{C}, A)$ is fully faithful and, if \mathcal{C} is essentially small, then $\mathcal{F}_{*, \mathcal{C}}: \text{L}_R(\mathcal{C}, A) \longrightarrow \text{QCoh}(\mathcal{X} \times A)$ is exact and the natural map $\mathcal{G} \longrightarrow \mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}}$ is an isomorphism.*

If X is quasi-projective then it is a classical fact that $\mathcal{C}_X = \{\mathcal{O}_X(n)\}_{n \in \mathbb{Z}}$ generates $\text{QCoh } X$ and thus we recover the classical properties of $\Gamma_*: \text{QCoh}(X \times A) \longrightarrow \text{GMod}(S_X \otimes_R A)$. Theorem above when \mathcal{C} consists of a single object is a rephrasing of classical Gabriel-Popescu's theorem for the category $\text{QCoh } \mathcal{X}$ (see 3.19). When \mathcal{C} is monoidal and generates $\text{QCoh } \mathcal{X}$ we also have that $\Omega^*: \text{QAlg}(\mathcal{X} \times A) \longrightarrow \text{ML}_R(\mathcal{C}, A)$ is fully faithful (see 3.34).

The second problem we address is to describe the essential image of Ω^* . The main idea behind this description is that $\text{Hom}_{\mathcal{X}}(-, \mathcal{F})$ for $\mathcal{F} \in \text{QCoh } \mathcal{X}$ is a left exact functor. Since the domain \mathcal{C} of the functors $\Omega^{\mathcal{F}}$ is not abelian we need an ad hoc definition of exactness. A *test sequence* in \mathcal{C} is an exact sequence

$$\mathcal{T}_*: \bigoplus_{k \in K} \mathcal{E}_k \xrightarrow{\alpha} \bigoplus_{i \in I} \mathcal{E}_i \longrightarrow \mathcal{E} \longrightarrow 0 \text{ where } \mathcal{E}, \mathcal{E}_i, \mathcal{E}_k \in \mathcal{C} \text{ and } I, J \text{ are sets}$$

such that $\alpha(\mathcal{E}_k)$ is contained in a finite sum for all $k \in K$. A test sequence is called *finite* if K and I are finite sets. Given $\Gamma \in \text{L}_R(\mathcal{C}, A)$ we will say that Γ is exact on a test sequence \mathcal{T}_* in \mathcal{C} if the complex of A -modules (see 3.9)

$$0 \longrightarrow \Gamma_{\mathcal{E}} \longrightarrow \prod_{i \in I} \Gamma_{\mathcal{E}_i} \longrightarrow \prod_{k \in K} \Gamma_{\mathcal{E}_k}$$

is exact. We denote by $\text{Lex}_R(\mathcal{C}, A)$ the full subcategory of $\text{L}_R(\mathcal{C}, A)$ of functors which are exact on all test sequences. We have the following:

Theorem (3.18, 3.26). *Let \mathcal{X} be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C} \subseteq \text{QCoh } \mathcal{X}$ be a full subcategory generating $\text{QCoh } \mathcal{X}$. Then $\text{Lex}_R(\mathcal{C}, A)$ is the essential image of the (fully faithful) functor $\Omega^*: \text{QCoh}(\mathcal{X} \times A) \rightarrow \text{L}_R(\mathcal{C}, A)$. If \mathcal{X} is quasi-compact and all sheaves in \mathcal{C} are finitely presented then $\text{Lex}_R(\mathcal{C}, A)$ is the subcategory of $\text{L}_R(\mathcal{C}, A)$ of functors which are exact on finite test sequences.*

In particular, when \mathcal{C} is essentially small, $\Omega^*: \text{QCoh}(\mathcal{X} \times A) \rightarrow \text{Lex}_R(\mathcal{C}, A)$ and $\mathcal{F}_{*, \mathcal{C}}: \text{Lex}_R(\mathcal{C}, A) \rightarrow \text{QCoh}(\mathcal{X} \times A)$ are quasi-inverses of each other. The two theorems above apply in the following situations (see 3.29, 3.30 and 3.31): if $\mathcal{C} = \text{QCoh } \mathcal{X}$ then $\text{Lex}_R(\text{QCoh } \mathcal{X}, A)$ is the category of contravariant, R -linear and left exact functors $\text{QCoh } \mathcal{X} \rightarrow \text{Mod } A$ which transform direct sums into products, if $\mathcal{C} = \text{Coh}(\mathcal{X})$ (resp. $\mathcal{C} = \text{Loc}(\mathcal{X})$) and \mathcal{X} is a noetherian algebraic stack (resp. \mathcal{X} is quasi-compact and has the resolution property) then $\text{Lex}_R(\mathcal{C}, A)$ is the category of contravariant, R -linear and left exact functors $\mathcal{C} \rightarrow \text{Mod } A$.

When \mathcal{C} is essentially small there is another cohomological characterization of the functors in $\text{Lex}_R(\mathcal{C}, A)$. A collection of maps $\mathcal{U} = \{\mathcal{E}_i \rightarrow \mathcal{E}\}$ in \mathcal{C} is called jointly surjective if the map $\bigoplus_{i \in I} \mathcal{E}_i \rightarrow \mathcal{E}$ is surjective. Given such a collection \mathcal{U} we set $\Delta_{\mathcal{U}} = \text{Im}(\bigoplus_{i \in I} \Omega^{\mathcal{E}_i} \rightarrow \Omega^{\mathcal{E}}) \in \text{L}_R(\mathcal{C}, R)$. Denote by \mathcal{C}^{\oplus} the subcategory of $\text{QCoh } \mathcal{X}$ consisting of all possible finite direct sums of sheaves in \mathcal{C} . We have:

Theorem (3.24, 3.26 and 3.28). *Let \mathcal{X} be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C} \subseteq \text{QCoh } \mathcal{X}$ be a full and essentially small subcategory generating $\text{QCoh } \mathcal{X}$. Then $\text{Lex}_R(\mathcal{C}, A)$ is the full subcategory of $\text{L}_R(\mathcal{C}, A)$ of functors Γ satisfying*

$$\text{Hom}_{\text{L}_R(\mathcal{C}, R)}(\Omega^{\mathcal{E}} / \Delta_{\mathcal{U}}, \Gamma) = \text{Ext}_{\text{L}_R(\mathcal{C}, R)}^1(\Omega^{\mathcal{E}} / \Delta_{\mathcal{U}}, \Gamma) = 0$$

for all jointly surjective collections of maps $\mathcal{U} = \{\mathcal{E}_i \rightarrow \mathcal{E}\}_{i \in I}$ in \mathcal{C} . If \mathcal{X} is quasi-compact and the sheaves in \mathcal{C} are finitely presented we can consider only finite collections \mathcal{U} .

We have $\text{Lex}_R(\mathcal{C}^{\oplus}, A) \simeq \text{Lex}_R(\mathcal{C}, A)$ via the restriction $\mathcal{C} \rightarrow \mathcal{C}^{\oplus}$ and, if \mathcal{C} is additive and \mathcal{J} is the smallest Grothendieck topology on \mathcal{C} containing the sieves $\Delta_{\mathcal{U}}$ for all jointly surjective collections $\mathcal{U} = \{\mathcal{E}_i \rightarrow \mathcal{E}\}_{i \in I}$ in \mathcal{C} , then $\text{Lex}_R(\mathcal{C}, A)$ coincides with the category of sheaves of A -modules $\mathcal{C}^{\text{op}} \rightarrow \text{Mod } A$ on the site $(\mathcal{C}, \mathcal{J})$ which are R -linear.

Besides Tannaka's reconstruction problem, theory above has two other applications. The first, which is also the original motivation, is the theory of Galois cover. More precisely in my Ph.D. thesis [Ton13b] I have worked out theory above in the case $\mathcal{X} = \text{B } G$ and $\mathcal{C} = \text{Loc } \mathcal{X}$, where G is a finite, flat and finitely presented group scheme over R . Notice that $\text{B } G$ satisfies the resolution property in this case (see 4.3). The proof presented in [Ton13b] makes use of representation theory and can not be generalized to arbitrary categories fibered in groupoids. The goal was to look at Galois covers with group G as particular monoidal functors, as G -torsors can be thought of as particular strong monoidal functors, and the motivation was the study of non-abelian Galois covers, where a direct approach as in the abelian case (see [Ton13a]) fails due to the complexity of the representation theory.

A second application is to the theory of Cox rings and homogeneous sheafifications. The idea is to consider $\mathcal{C}_H = \{\mathcal{L}\}_{\mathcal{L} \in H} \subseteq \text{Loc } \mathcal{X}$ where H is a subgroup of $\text{Pic } \mathcal{X}$. As in the projective case we have a homogeneous coordinate ring

$$S_H = \bigoplus_{\mathcal{L} \in H} H^0(\mathcal{X}, \mathcal{L})$$

(opportunately defined), $L_R(\mathcal{C}_H, A)$ is equivalent to $\mathrm{GMod}(S_H \otimes_R A)$, the category of H -graded $(S_H \otimes_R A)$ -modules, Ω^* corresponds to

$$\Gamma_*: \mathrm{QCoh}(\mathcal{X} \times A) \longrightarrow \mathrm{GMod}(S_H \otimes_R A), \quad \Gamma_*(\mathcal{F}) = \bigoplus_{\mathcal{L} \in H} H^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{L})$$

and its adjoint $\mathcal{F}_{*, \mathcal{C}_H}$ behaves like a homogeneous sheafification. Moreover in more concrete geometric situations, e.g. when \mathcal{X} is a normal variety, there are analogous constructions for reflexive sheaves of rank 1. We expect that this theory covers all known cases where Γ_* is proved to be fully faithful (see for instance [CLS11, Appendix of Chapter 6] and [Kaj98, Section 2]).

Applications above are not described in the present paper, but, hopefully, they will be subjects of future ones.

The outline of the paper is the following. The first section introduces the notion and the basic properties of quasi-coherent sheaves on fibered categories, while the second one is a general study of sheafification functors. In the third section we study the fully faithfulness and the essential image of the functor $\Omega^*: \mathrm{QCoh}(\mathcal{X} \times A) \longrightarrow L_R(\mathcal{C}, A)$. In the fourth section we rewrite the results obtained in the case of the stack of G -torsor in terms of the representation theory of G and finally, in the last section, we prove the non-neutral Tannaka's reconstruction.

NOTATION

In this paper we work over a base commutative, associative ring R with unity. If not stated otherwise a fiber category will be a category fibered in groupoids over Aff/R , the category of affine schemes over $\mathrm{Spec} R$, or, equivalently, the opposite of the category of R -algebras. A scheme or an algebraic space X over $\mathrm{Spec} R$ will be thought of as the fibered category of maps from an affine scheme to X , denoted by Aff/X . A map or morphism of fibered categories is a functor over Aff/R . Recall that by the 2-Yoneda lemma objects of a fibered category \mathcal{X} can be thought of as maps $T \longrightarrow \mathcal{X}$ from an affine scheme. An fpqc stack will be a stack for the fpqc topology.

A map $f: \mathcal{X}' \longrightarrow \mathcal{X}$ of fibered categories is called representable if for all maps $T \longrightarrow \mathcal{X}$ from an affine scheme (or an algebraic space) the fiber product $T \times_{\mathcal{X}} \mathcal{X}'$ is (equivalent to) an algebraic space.

Given a flat and affine group scheme G over T we denote by $B_R G$ the stack of G -torsors for the fpqc topology, which is an fpqc stack with affine diagonal. When $G \longrightarrow \mathrm{Spec} R$ is finitely presented (resp. smooth) then $B_R G$ coincides with the stack of G -torsors for the fppf (resp. étale) topology.

By a “subcategory” of a given category we mean a “full subcategory” if not stated otherwise.

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1. PRELIMINARIES ON SHEAVES AND FIBERED CATEGORIES

Let $\pi: \mathcal{X} \longrightarrow \mathrm{Aff}/R$ be a fibered category. There is a functor of rings $\mathcal{O}_{\mathcal{X}}: \mathcal{X}^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$ defined by $\mathcal{O}_{\mathcal{X}}(\xi) = H^0(\mathcal{O}_{\pi(\xi)})$, so that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a ringed category. A presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on \mathcal{X} is a functor $\mathcal{F}: \mathcal{X}^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$ together with an $H^0(\mathcal{O}_{\pi(\xi)})$ -module structure on $\mathcal{F}(\xi)$ for all $\xi \in \mathcal{X}$ such that, if $\xi \longrightarrow \eta$ is a map on \mathcal{X} , then the map $\mathcal{F}(\eta) \longrightarrow \mathcal{F}(\xi)$ is $H^0(\mathcal{O}_{\pi(\eta)})$ -linear. Morphism of presheaves of $\mathcal{O}_{\mathcal{X}}$ -modules are natural transformations respecting the module structures. We denote by $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ the category of presheaves of $\mathcal{O}_{\mathcal{X}}$ -modules.

Definition 1.1. A quasi-coherent sheaf over \mathcal{X} is a presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules such that for all maps $\xi \rightarrow \eta$ in \mathcal{X} the induced map

$$\mathcal{F}(\eta) \otimes_{H^0(\mathcal{O}_{\pi(\eta)})} H^0(\mathcal{O}_{\pi(\xi)}) \rightarrow \mathcal{F}(\xi)$$

is an isomorphism. We denote by $\mathrm{QCoh} \mathcal{X}$ the full subcategory of $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ of quasi-coherent sheaves.

Notice that by fpqc descent of modules a quasi-coherent sheaf is a sheaf for the fpqc topology of \mathcal{X} .

If $f: \mathcal{Y} \rightarrow \mathcal{X}$ is a morphism of fibered categories and $\mathcal{F} \in \mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ we define $f^* \mathcal{F} = \mathcal{F} \circ f: \mathcal{Y}^{\mathrm{op}} \xrightarrow{f} \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathcal{F}} (\mathrm{Sets})$. This association defines a functor $f^*: \mathrm{Mod} \mathcal{O}_{\mathcal{X}} \rightarrow \mathrm{Mod} \mathcal{O}_{\mathcal{Y}}$, called the pull-back functor, and restricts to a functor $f^*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh} \mathcal{Y}$. Notice that $f^* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ tautologically.

The category $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ is an abelian category and cokernels of maps between quasi-coherent sheaves are again quasi-coherent and thus cokernels in $\mathrm{QCoh} \mathcal{X}$. Notice that, since we have restricted our fiber categories to affine schemes, a map $f: \mathcal{F} \rightarrow \mathcal{G}$ of quasi-coherent sheaves is an epimorphism if and only if it is pointwise surjective, that is surjective in $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$. The category $\mathrm{QCoh} \mathcal{X}$ is R -linear but it is unclear if it is abelian. Moreover kernels in $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ of maps between quasi-coherent sheaves are almost never quasi-coherent, essentially because pull-backs are not left exact. There is a natural condition on \mathcal{X} which allows us to prove that $\mathrm{QCoh} \mathcal{X}$ is an R -linear abelian category.

Definition 1.2. An *fpqc atlas* (or simply *atlas*) of a fibered category \mathcal{X} is a representable fpqc covering $X \rightarrow \mathcal{X}$ from a scheme. A fiber category is called *pseudo-algebraic* if it has an atlas, it is called *quasi-compact* if it has an atlas from an affine scheme.

Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of fibered categories. The map f is called *pseudo-algebraic* (resp. *quasi-compact*) if for all maps $T \rightarrow \mathcal{X}$ from a scheme (resp. quasi-compact scheme) the fiber product $T \times_{\mathcal{X}} \mathcal{Y}$ is pseudo-algebraic (resp. quasi-compact). It is called *quasi-separated* if the diagonal $\mathcal{Y} \rightarrow \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is quasi-compact.

If \mathcal{X} is pseudo-algebraic then the diagonal $\Delta_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X} \times_R \mathcal{X}$ is not necessarily representable. It is unclear whether this is true or not if \mathcal{X} is a fpqc stack, because it is not known if algebraic spaces satisfy effective descent along fpqc coverings.

Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a map of fibered categories. If \mathcal{X} and f are pseudo-algebraic then \mathcal{Y} is pseudo-algebraic. If \mathcal{Y} is pseudo-algebraic and $\Delta_{\mathcal{X}}$ is representable then f is pseudo-algebraic.

If \mathcal{C} is a (not full) subcategory of \mathcal{X} one can analogously define presheaves of $(\mathcal{O}_{\mathcal{X}})_{|\mathcal{C}}$ -modules and quasi-coherent sheaves on \mathcal{C} just replacing all occurrences of \mathcal{X} with \mathcal{C} . We denote by $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})_{|\mathcal{C}}$ and $\mathrm{QCoh} \mathcal{C}$ the resulting categories.

Definition 1.3. We define $\mathcal{X}_{\mathrm{fl}}$ (resp. $\mathcal{X}_{\mathrm{sm}}$, $\mathcal{X}_{\mathrm{et}}$) as the (not full) subcategory of \mathcal{X} of objects $\xi: \mathrm{Spec} B \rightarrow \mathcal{X}$ which are representable and flat (resp. smooth, étale) and the arrows are morphisms in \mathcal{X} whose underlying map of affine schemes is flat (resp. smooth, étale).

If $X \rightarrow \mathcal{X}$ is an fpqc atlas then by definition $R = X \times_{\mathcal{X}} X$ is an algebraic space and the two projections $R \rightrightarrows X$ extends to a groupoid in algebraic spaces. We denote by $\mathrm{QCoh}(R \rightrightarrows X)$ the category of quasi-coherent sheaves on $R \rightrightarrows X$ (see [SP014, Tag 0440]). By standard arguments of fpqc descent for modules we have:

Proposition 1.4. *If \mathcal{X} admits an fpqc (resp. smooth, étale) atlas then the restriction $\mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh} \mathcal{X}_{\mathrm{fl}}$ (resp. $\mathrm{QCoh} \mathcal{X}_{\mathrm{sm}}$, $\mathrm{QCoh} \mathcal{X}_{\mathrm{et}}$) is an equivalence of categories. If $f: X \rightarrow \mathcal{X}$ is an fpqc atlas then $f^*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh} X$ is faithful and it induces an equivalence $\mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh}(R \rightrightarrows X)$.*

We see that if \mathcal{X} is pseudo-algebraic then $\mathrm{QCoh} \mathcal{X}$ is equivalent to an R -linear abelian category, namely $\mathrm{QCoh}(R \rightrightarrows X)$. Moreover if $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ is a map of quasi-coherent sheaves then $\mathrm{Ker}(\alpha)$ is defined by taking $\mathrm{Ker}(\alpha|_{\mathcal{X}_\eta}) \in \mathrm{QCoh} \mathcal{X}_\eta$, which is just given by $\mathrm{Ker}(\alpha|_{\mathcal{X}_\eta})(\mathrm{Spec} B \xrightarrow{\xi} \mathcal{X}) = \mathrm{Ker}(\alpha(\xi): \mathcal{F}(\xi) \rightarrow \mathcal{G}(\xi))$ for $\xi \in \mathcal{X}_\eta$, and then extending it to the whole \mathcal{X} . If \mathcal{X} is an algebraic stack or a scheme we see that $\mathrm{QCoh} \mathcal{X}$ is equivalent to the usual category of quasi-coherent sheaves via an R -linear and exact functor. If it is given a subcategory \mathcal{D} of $\mathrm{QCoh} \mathcal{X}$, an exact sequence of sheaves in \mathcal{D} will always be an exact sequence in $\mathrm{QCoh} \mathcal{X}$ of sheaves belonging to \mathcal{D} .

We now deal with the problem of defining a right adjoint of a pull-back functor. Given $\mathcal{F} \in \mathrm{Mod} \mathcal{O}_\mathcal{X}$ we define the global section $\mathcal{F}(\mathcal{X}) = \mathrm{Hom}(\mathcal{O}_\mathcal{X}, \mathcal{F})$ of \mathcal{F} , also denoted by $H^0(\mathcal{X}, \mathcal{F})$ or simply $H^0(\mathcal{F})$, which is an $\mathcal{O}_\mathcal{X}(\mathcal{X})$ -module. More generally given a map of fibered categories $g: \mathcal{Z} \rightarrow \mathcal{X}$ we define $\mathcal{F}(\mathcal{Z}) = (g^*\mathcal{F})(\mathcal{Z})$. If $\mathcal{Z} = \mathrm{Spec} B$ is affine we will often write $\mathcal{F}(B)$ instead of $\mathcal{F}(\mathrm{Spec} B)$.

Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a map of fibered categories. Given $\mathcal{G} \in \mathrm{Mod} \mathcal{O}_\mathcal{Y}$ and an object $\xi: T \rightarrow \mathcal{X}$ of \mathcal{X} we define

$$(f_p\mathcal{G})(\xi) = \mathcal{G}(T \times_\mathcal{X} \mathcal{Y})$$

Given another object $\xi': T' \rightarrow \mathcal{X}$ and a morphism $\beta: \xi' \rightarrow \xi$ in \mathcal{X} there is an induced morphism $(f_p\mathcal{G})(\xi) \rightarrow (f_p\mathcal{G})(\xi')$. This data define a functor $f_p: \mathrm{Mod} \mathcal{O}_\mathcal{Y} \rightarrow \mathrm{Mod} \mathcal{O}_\mathcal{X}$ and we have:

Proposition 1.5. *Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a map of fibered categories. Then f_p is a right adjoint of f^* and, if*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow f' & & \downarrow f \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \end{array}$$

is a 2-cartesian diagram of fibered categories, there is an isomorphism of functors

$$g^*f_p \rightarrow f'_p g'^*: \mathrm{Mod} \mathcal{O}_\mathcal{Y} \rightarrow \mathrm{Mod} \mathcal{O}_{\mathcal{X}'}$$

If f is affine then $f_p(\mathrm{QCoh} \mathcal{Y}) \subseteq \mathrm{QCoh} \mathcal{X}$ and $(f_p)|_{\mathrm{QCoh} \mathcal{Y}}: \mathrm{QCoh} \mathcal{Y} \rightarrow \mathrm{QCoh} \mathcal{X}$ is right adjoint to $f^: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh} \mathcal{Y}$.*

Proof. The adjunction between f^* and f_p can be found in [SP014, Tag 00XF]. With notation from this reference, we have that $f_\eta^* \simeq T \times_\mathcal{X} \mathcal{Y}$ for an object $\eta: T \rightarrow \mathcal{X}$. Moreover if $\mathcal{F} \in \mathrm{Mod} \mathcal{O}_\mathcal{Y}$ then $\mathcal{F}(T \times_\mathcal{X} \mathcal{Y})$ is the limit of $\mathcal{F}|_{T \times_\mathcal{X} \mathcal{Y}}$ over the whole $T \times_\mathcal{X} \mathcal{Y}$. Thus what is denoted by $f_p\mathcal{F}$ is easily seen to be equivalent to our f_p if we take limits of R -modules and not of sets. The isomorphism for the base change is tautological. For the last claim we can assume that \mathcal{X} is an affine scheme in which case the result follows because (usual) push-forwards commutes with arbitrary base changes. \square

In general f_p does not preserve quasi-coherent sheaves, even if f is a proper map of schemes. To get a right adjoint of pullback we have to require more.

Definition 1.6. A pseudo-algebraic map $f: \mathcal{Y} \rightarrow \mathcal{X}$ of fibered categories is called *flat* if given an object $\xi: T \rightarrow \mathcal{X}$ of \mathcal{X} and an atlas $V \rightarrow T \times_\mathcal{X} \mathcal{Y}$ the resulting map $V \rightarrow T$ is flat.

Proposition 1.7. *Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a quasi-compact and quasi-separated map of pseudo-algebraic fibered categories. Then the composition $\mathrm{QCoh} \mathcal{Y} \xrightarrow{f_p} \mathrm{Mod} \mathcal{O}_\mathcal{X} \rightarrow \mathrm{Mod}(\mathcal{O}_\mathcal{X})|_{\mathcal{X}_\eta}$ has values in $\mathrm{QCoh} \mathcal{X}_\eta$. The induced map $f_*: \mathrm{QCoh} \mathcal{Y} \rightarrow \mathrm{QCoh} \mathcal{X}$ is a right adjoint of*

$f^*: \mathrm{QCoh} \mathcal{X} \longrightarrow \mathrm{QCoh} \mathcal{Y}$. If

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow f' & & \downarrow f \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \end{array}$$

is a 2-cartesian diagram of fibered categories with \mathcal{X}' pseudo-algebraic then \mathcal{Y}' is pseudo-algebraic, f' is quasi-compact and quasi-separated and there is a natural transformation of functors

$$g^* f_* \longrightarrow f'_* g'^*: \mathrm{QCoh} \mathcal{X} \longrightarrow \mathrm{QCoh} \mathcal{Y}'$$

which is an isomorphism if g is flat.

Proof. Consider the 2-Cartesian diagram in the statement. The diagonal of f' is quasi-compact because it is base change of the diagonal of f . To see that $f_p(\mathcal{F})|_{\mathcal{X}'_p}$ is quasi-coherent for $\mathcal{F} \in \mathrm{QCoh} \mathcal{Y}$, we can assume $\mathcal{X} = \mathrm{Spec} B$ affine and that \mathcal{Y} is quasi-compact with quasi-compact diagonal. If $U = \mathrm{Spec} A \longrightarrow \mathcal{Y}$ is a fpqc atlas, it follows that $R = U \times_{\mathcal{Y}} U$ is a quasi-compact algebraic space. By covering R by finitely many affine schemes $\mathrm{Spec} A_i$ we can write $\mathcal{F}(\mathcal{Y})$ as kernel of a map $\mathcal{F}(A) \longrightarrow \bigoplus_i \mathcal{F}(A_i)$. If we base change along a flat map $B \longrightarrow B'$ it is now easy to see that $\mathcal{F}(\mathcal{Y} \times_B B') \simeq \mathcal{F}(\mathcal{Y}) \otimes_B B'$, as required.

To define the natural transformation $\alpha: g^* f_* \longrightarrow f'_* g'^*$ notice that there is a natural map $f_* \mathcal{F} \longrightarrow f_p \mathcal{F}$ which extends the identity on \mathcal{X}_p . Applying g^* we get $g^* f_* \mathcal{F} \longrightarrow g^* f_p \mathcal{F} \simeq f'_p g'^* \mathcal{F}$ and then, restricting to \mathcal{X}'_p , a map $(g^* f_* \mathcal{F})|_{\mathcal{X}'_p} \longrightarrow (f'_p g'^* \mathcal{F})|_{\mathcal{X}'_p}$. Since both sides are in $\mathrm{QCoh} \mathcal{X}'_p$ this map uniquely extends to a natural transformation α as required. Finally assume that g is flat and let $\xi: \mathrm{Spec} B \longrightarrow \mathcal{X}' \in \mathcal{X}'_p$. If the composition $\mathrm{Spec} B \longrightarrow \mathcal{X}$ is in \mathcal{X}_p then one can easily check that $\alpha(\xi)$ is an isomorphism. Otherwise, by definition of flatness, there exists an fpqc covering $\mathrm{Spec} B' \longrightarrow \mathrm{Spec} B$ whose composition $\xi': \mathrm{Spec} B' \longrightarrow \mathcal{X}'$ satisfies the previous condition. Since $\alpha(\xi) \otimes_B B' \simeq \alpha(\xi')$ we get the desired result. \square

Remark 1.8. There are set-theoretic problems in considering global sections of presheaves and therefore push-forwards, because $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ is in general not locally small. The common way to solve this problem is to use Grothendieck universes. Take a universe U and define rings inside U , so that Aff/R is small (with respect to a bigger universe). Fibered categories should then be required to be small too. In this situation it is easy to show that $\mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ is locally small and therefore global sections and push-forwards are well defined. With this approach we have to be careful in considering (big) rings defined starting from some $\mathcal{F} \in \mathrm{Mod} \mathcal{O}_{\mathcal{X}}$: for instance $\mathrm{Spec} \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ is in general not an object of Aff/R .

Notice that global sections and pushforwards of quasi-coherent sheaves are always well defined for a pseudo-algebraic fibered category and a pseudo-algebraic map respectively. The reason is that if $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$ and $p: X \longrightarrow \mathcal{X}$ is a fpqc atlas then $\mathcal{F}(\mathcal{X}) \longrightarrow (p^* \mathcal{F})(X)$ is injective and thus $\mathcal{F}(\mathcal{X})$ is a set.

In the rest of the paper we will not be concerned about those set-theoretic problems.

Definition 1.9. If A is an R -algebra and $\mathcal{F} \in \mathrm{Mod} \mathcal{O}_{\mathcal{X}}$ then a compatible A -module structure on \mathcal{F} is the data of A -module structures on $\mathcal{F}(\xi)$ commuting with the $H^0(\mathcal{O}_{\pi(\xi)})$ -module structure on $\mathcal{F}(\xi)$ for all $\xi \in \mathcal{X}$ and such that, for all $\xi \longrightarrow \eta$ in \mathcal{X} , the map $\mathcal{F}(\eta) \longrightarrow \mathcal{F}(\xi)$ is A -linear. We define $\mathrm{QCoh}_A \mathcal{X}$ as the category of quasi-coherent sheaves over \mathcal{X} with an A -module structure. We also define \mathcal{X}_A as the fiber product $\mathrm{Spec} A \times_R \mathcal{X}$.

Notice that if $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ is a map of fibered categories and \mathcal{F} is a presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules with an A -module structure then $g^* \mathcal{F}$ inherits an A -module structure. In particular $g^*: \mathrm{QCoh} \mathcal{X} \longrightarrow \mathrm{QCoh} \mathcal{Y}$ extends to a functor $\mathrm{QCoh}_A \mathcal{X} \longrightarrow \mathrm{QCoh}_A \mathcal{Y}$.

Proposition 1.10. *Let A be an R -algebra. Then the push-forward map $\mathrm{QCoh} \mathcal{X}_A \rightarrow \mathrm{QCoh} \mathcal{X}$ extends naturally to an equivalence $\mathrm{QCoh} \mathcal{X}_A \rightarrow \mathrm{QCoh}_A \mathcal{X}$.*

Proof. The result is very simple if \mathcal{X} is an affine scheme. In general, if we set $g: \mathcal{X}_A \rightarrow \mathcal{X}$ for the projection and we consider $\mathcal{G} \in \mathrm{QCoh} \mathcal{X}_A$, then $g_* \mathcal{G} \in \mathrm{QCoh} \mathcal{X}$ and it inherits an A -module structure from the action of A on \mathcal{G} . Therefore $g_* \mathcal{G} \in \mathrm{QCoh}_A \mathcal{X}$. If $h: \mathrm{Spec} B \rightarrow \mathcal{X}$ is a map consider the diagrams

$$\begin{array}{ccc} \mathrm{Spec}(B \otimes_R A) & \xrightarrow{h'} & \mathcal{X}_A \\ \downarrow g' & & \downarrow g \\ \mathrm{Spec} B & \xrightarrow{h} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} \mathrm{QCoh} \mathcal{X}_A & \xrightarrow{h'^*} & \mathrm{QCoh} \mathrm{Spec}(B \otimes_R A) \\ \downarrow g_* & & \downarrow g'_* \\ \mathrm{QCoh}_A \mathcal{X} & \xrightarrow{h^*} & \mathrm{QCoh}_A \mathrm{Spec} B \end{array}$$

The second diagram is 2-commutative and the last vertical map is an equivalence. Using those diagrams it is easy to define a quasi-inverse $\mathrm{QCoh}_A \mathcal{X} \rightarrow \mathrm{QCoh} \mathcal{X}_A$ of g_* . \square

We will almost always regard quasi-coherent sheaves over \mathcal{X}_A as objects of $\mathrm{QCoh}_A \mathcal{X}$.

Remark 1.11. If $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}_A \mathcal{X}$ then $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ does not correspond to the tensor product in $\mathrm{QCoh} \mathcal{X}_A$: $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ has two distinct A -module structures. Under the equivalence $\mathrm{QCoh} \mathcal{X}_A \rightarrow \mathrm{QCoh}_A \mathcal{X}$ the tensor product of \mathcal{F} and \mathcal{G} , that we will denote by $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{G}$, is given by

$$U \mapsto \mathcal{F}(U) \otimes_{H^0(\mathcal{O}_U)} \mathcal{G}(U) / \langle ax \otimes y - x \otimes ay \mid x \in \mathcal{F}(U), y \in \mathcal{G}(U) \rangle$$

Definition 1.12. A locally free sheaf \mathcal{E} (of rank n) over \mathcal{X} is a quasi-coherent sheaf such that $\mathcal{E}(\mathrm{Spec} B \rightarrow \mathcal{X})$ is a finitely generated projective B -module (of rank n) for all maps $\mathrm{Spec} B \rightarrow \mathcal{X}$. We denote by $\mathrm{Loc} \mathcal{X}$ the subcategory of $\mathrm{QCoh} \mathcal{X}$ of locally free sheaves.

We will say that a fiber category \mathcal{X} has the *resolution property* if $\mathrm{Loc} \mathcal{X}$ generates $\mathrm{QCoh} \mathcal{X}$.

2. SHEAFIFICATION FUNCTORS.

In this section we define and describe particular functors that generalize sheafification functors for affine schemes or projective schemes. The idea is to interpret the category of modules or graded modules respectively as a category of R -linear functors. More precisely:

Definition 2.1. Given a fibered category \mathcal{X} over a ring R , an R -algebra A and a subcategory \mathcal{D} of $\mathrm{QCoh} \mathcal{X}$ we define $L_R(\mathcal{D}, A)$ as the category of contravariant R -linear functors $\Gamma: \mathcal{D} \rightarrow \mathrm{Mod} A$ and natural transformations as arrows. We define a functor $\Omega^*: \mathrm{QCoh}_A \mathcal{X} \rightarrow L_R(\mathcal{D}, A)$ by

$$\Omega^{\mathcal{F}} = \mathrm{Hom}_{\mathcal{X}}(-, \mathcal{F}): \mathcal{D} \rightarrow \mathrm{Mod} A$$

The functor Ω^* is called the *Yoneda functor* associated with \mathcal{D} . A left adjoint of Ω^* is called a *sheafification* functor associated with \mathcal{D} . If $\mathcal{F} \in \mathrm{QCoh}_A \mathcal{X}$ we will call $\Omega^{\mathcal{F}}$ the Yoneda functor associated with \mathcal{F} .

Example 2.2. The analogy with the sheafification functor associated with $\mathcal{X} = \mathbb{P}_R^n$ or any quasi-projective and quasi-compact scheme over R is the following. If $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}(n)\}_{n \in \mathbb{Z}}$ then with $\Gamma \in L_R(\mathcal{C}, R)$ we can associate the \mathbb{Z} -graded R -module $M = \bigoplus_{n \in \mathbb{Z}} \Gamma_{\mathcal{O}_{\mathcal{X}}(-n)}$. The functorial properties of Γ allow us to define a structure of graded S -module on M , where $S = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_{\mathcal{X}}(n))$ is the homogeneous coordinate ring of \mathcal{X} . This associations extends to an equivalence of categories between $L_R(\mathcal{C}, R)$ and the category of graded S -modules. The functor Ω^* corresponds to the functor Γ_* which carries a quasi-coherent sheaf \mathcal{F} on \mathcal{X} to the graded S -module $\bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{F}(n))$. Finally the sheafification functor from the graded S -modules to $\mathrm{QCoh} \mathcal{X}$ is left adjoint to Γ_* .

Let us fix an R -algebra A and a fibered category $\pi: \mathcal{X} \rightarrow \mathrm{Aff}/R$.

2.1. Sheafifying R -linear functors. In this section we want to explicitly describe sheafification functors for small subcategories of $\mathrm{QCoh} \mathcal{X}$. In particular we fix a *small* (and non empty) subcategory \mathcal{C} of $\mathrm{QCoh} \mathcal{X}$.

In the construction of the sheafification functors we will make use of the coend construction in the settings of categories enriched over categories of modules over a ring. The general theory simplifies considerably in this context and we will also apply such construction only in particular cases. In the following remark we collect all the properties we will need.

Remark 2.3. Let \mathcal{Y} be a fibered category over R , $F: \mathcal{C} \rightarrow \mathrm{QCoh} \mathcal{Y}$ be an R -linear functor and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$. The coend of the R -linear functor $\Gamma_- \otimes_R F_-: \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \rightarrow \mathrm{QCoh}_A \mathcal{Y}$, denoted by

$$\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \in \mathrm{QCoh}_A \mathcal{Y}$$

is the cokernel of the map

$$\bigoplus_{\mathcal{E} \xrightarrow{u} \bar{\mathcal{E}}} (\Gamma_u \otimes \mathrm{id}_{F_{\mathcal{E}}} - \mathrm{id}_{\Gamma_{\bar{\mathcal{E}}}} \otimes F_u): \bigoplus_{\mathcal{E} \xrightarrow{u} \bar{\mathcal{E}}} \Gamma_{\bar{\mathcal{E}}} \otimes_R F_{\mathcal{E}} \rightarrow \bigoplus_{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$$

Moreover it comes equipped with an A -linear natural isomorphism

$$\mathrm{Hom}_{\mathrm{QCoh}_A \mathcal{Y}} \left(\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}, \mathcal{H} \right) \xrightarrow{\alpha} \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, A)} (\Gamma, \mathrm{Hom}_{\mathcal{Y}}(F_-, \mathcal{H})) \text{ for } \mathcal{H} \in \mathrm{QCoh}_A \mathcal{Y}$$

given by

$$\alpha \left(\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \xrightarrow{u} \mathcal{H} \right) (x) = u \circ p_{\bar{\mathcal{E}}}(x \otimes -): F_{\bar{\mathcal{E}}} \rightarrow \mathcal{H} \text{ for } \bar{\mathcal{E}} \in \mathcal{C}, x \in \Gamma_{\bar{\mathcal{E}}}$$

where $p_{\mathcal{E}}: \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \rightarrow \int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{C}$ are the structure morphisms. Its inverse is uniquely determined by the expression

$$\alpha^{-1}(\Gamma \xrightarrow{v} \mathrm{Hom}_{\mathcal{Y}}(F_-, \mathcal{H})) \circ p_{\mathcal{E}}: \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \rightarrow \mathcal{H}, \quad x \otimes y \mapsto v_{\mathcal{E}}(x)(y) \text{ for } \mathcal{E} \in \mathcal{C}$$

Natural transformations $F \rightarrow F'$ and $\Gamma \rightarrow \Gamma'$ yields morphisms $\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \rightarrow \int^{\mathcal{E} \in \mathcal{C}} \Gamma'_{\mathcal{E}} \otimes_R F'_{\mathcal{E}}$ and $\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \rightarrow \int^{\mathcal{E} \in \mathcal{C}} \Gamma'_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ respectively. Those can be defined either using Yoneda's lemma and the above characterization of $\mathrm{Hom}(\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}, -)$ or directly using the description of $\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ as a cokernel.

All the above claims are standard in the theory of coend in the enriched settings (in our case enriched by $\mathrm{Mod} R$), but, in this simplified context, it is elementary to prove them directly.

We start by showing that \mathcal{C} (and therefore any essentially small subcategory of $\mathrm{QCoh} \mathcal{X}$) admits a sheafification functor.

Proposition 2.4. *The Yoneda functor $\Omega^*: \mathrm{QCoh}_A \mathcal{X} \rightarrow \mathrm{L}_R(\mathcal{C}, A)$ has a left adjoint $\mathcal{F}_{-, \mathcal{C}}: \mathrm{L}_R(\mathcal{C}, A) \rightarrow \mathrm{QCoh}_A \mathcal{X}$ given by*

$$\mathcal{F}_{\Gamma, \mathcal{C}}: \mathcal{X}^{\mathrm{op}} \rightarrow (\mathrm{Sets}), \quad \mathcal{X} \ni \xi \mapsto \int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathcal{E}(\xi) \in \mathrm{Mod}(\mathrm{H}^0(\mathcal{O}_{\pi(\xi)}) \otimes_R A)$$

where $\mathcal{E}(\xi)$ denotes the evaluation $\mathcal{C} \rightarrow \mathrm{Mod} \mathrm{H}^0(\mathcal{O}_{\pi(\xi)})$ of sheaves in $\xi \in \mathcal{X}$. Alternatively

$$\mathcal{F}_{\Gamma, \mathcal{C}} = \int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathcal{E} \in \mathrm{QCoh}_A \mathcal{X}$$

where \mathcal{E} denotes the inclusion $\mathcal{C} \rightarrow \mathrm{QCoh} \mathcal{X}$.

Proof. It is enough to apply 2.3 with $\mathcal{Y} = \mathcal{X}$ and $F: \mathcal{C} \rightarrow \text{QCoh } \mathcal{X}$ the inclusion. Using the description of coend as cokernel one can check that the two functors defined in the statement are canonically isomorphic. \square

Definition 2.5. We denote by $\gamma_\Gamma: \Gamma_- \rightarrow \Omega_-^{\mathcal{F}_\Gamma, \mathcal{C}} = \text{Hom}_{\mathcal{X}}(-, \mathcal{F}_{\Gamma, \mathcal{C}})$ and $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}} \rightarrow \mathcal{G}$ for $\Gamma \in \text{L}_R(\mathcal{C}, A)$ and $\mathcal{G} \in \text{QCoh}_A \mathcal{X}$ the unit and the counit of the adjunction between $\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{L}_R(\mathcal{C}, A)$ and $\mathcal{F}_{*, \mathcal{C}}: \text{L}_R(\mathcal{C}, A) \rightarrow \text{QCoh}_A \mathcal{X}$ respectively.

Given $\xi \in \mathcal{X}$, $\mathcal{E} \in \mathcal{C}$, $\psi \in \mathcal{E}(\xi)$ and $x \in \Gamma_{\mathcal{E}}$ we denote by $x_{\mathcal{E}, \psi} \in \mathcal{F}_{\Gamma, \mathcal{C}}(\xi)$ the image of $x \otimes \psi$ under the map $\Gamma_{\mathcal{E}} \otimes_R \mathcal{E}(\xi) \rightarrow \mathcal{F}_{\Gamma, \mathcal{C}}(\xi)$

Proposition 2.6. *Let $\Gamma \in \text{L}_R(\mathcal{C}, A)$. The unit $\gamma_\Gamma: \Gamma_- \rightarrow \Omega_-^{\mathcal{F}_\Gamma, \mathcal{C}} = \text{Hom}_{\mathcal{X}}(-, \mathcal{F}_{\Gamma, \mathcal{C}})$ is given by*

$$\begin{aligned} \Gamma_{\mathcal{E}} &\longrightarrow \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}_{\Gamma, \mathcal{C}}) \\ x &\longmapsto (\phi \mapsto x_{\mathcal{E}, \phi}) \end{aligned}$$

If $\mathcal{G} \in \text{QCoh}_A \mathcal{X}$ the counit $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}} \rightarrow \mathcal{G}$ is given by

$$\mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}}(\xi) \ni x_{\mathcal{E}, \psi} \longmapsto x(\psi) \in \mathcal{G}(\xi) \text{ for } \mathcal{E} \in \mathcal{C}, x \in \Omega_{\mathcal{E}}^{\mathcal{G}} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G}), \xi \in \mathcal{X}, \psi \in \mathcal{E}(\xi)$$

Proof. If $p_{\mathcal{E}}: \Gamma_{\mathcal{E}} \otimes_R \mathcal{E} \rightarrow \mathcal{F}_{\Gamma, \mathcal{C}}$ for $\mathcal{E} \in \mathcal{C}$ are the maps associated with the coend defining $\mathcal{F}_{\Gamma, \mathcal{C}}$, then $p_{\mathcal{E}}(x \otimes \psi) = x_{\mathcal{E}, \psi}$ for $x \in \Gamma_{\mathcal{E}}, \psi \in \mathcal{E}(\xi)$ and $\xi \in \mathcal{X}$. All the claims follows by a direct check using the explicit description of the isomorphism

$$\text{Hom}_{\text{QCoh}_A \mathcal{X}} \left(\int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathcal{E}, \mathcal{H} \right) \xrightarrow{\alpha} \text{Hom}_{\text{L}_R(\mathcal{C}, A)}(\Gamma, \Omega^{\mathcal{H}}) \text{ for } \mathcal{H} \in \text{QCoh}_A \mathcal{X}$$

and its inverse given in 2.3. \square

Given a map $g: \mathcal{Y} \rightarrow \mathcal{X}$ of fibered categories we want to express $g^* \mathcal{F}_{\Gamma, \mathcal{C}} \in \text{QCoh}_A \mathcal{Y}$ for $\Gamma \in \text{L}_R(\mathcal{C}, A)$ as $\mathcal{F}_{g^* \Gamma, g^* \mathcal{C}}$ for a suitable choice of $g^* \mathcal{C} \subseteq \text{QCoh } \mathcal{Y}$ and $g^* \Gamma \in \text{L}_R(g^* \mathcal{C}, A)$.

Definition 2.7. Let \mathcal{Y} be a fibered category, $g: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism and \mathcal{D} be a subcategory of $\text{QCoh } \mathcal{X}$. We set $g^* \mathcal{D}$ for the subcategory of $\text{QCoh } \mathcal{Y}$ of sheaves $g^* \mathcal{E}$ for $\mathcal{E} \in \mathcal{D}$. If $\mathcal{D}' \subseteq \text{QCoh } \mathcal{Y}$ is a subcategory containing $g^* \mathcal{D}$ we can define a restriction functor

$$\begin{aligned} \text{L}_R(\mathcal{D}', A) &\xrightarrow{g^*} \text{L}_R(\mathcal{D}, A) \\ \Gamma &\longmapsto \Gamma \circ g^* \end{aligned}$$

Proposition 2.8. *Let \mathcal{Y} be a fibered category, $g: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism and \mathcal{D} be a subcategory of $\text{QCoh } \mathcal{Y}$ such that $g^* \mathcal{C} \subseteq \mathcal{D}$. Then $g_*: \text{L}_R(\mathcal{D}, A) \rightarrow \text{L}_R(\mathcal{C}, A)$ has a left adjoint $g^*: \text{L}_R(\mathcal{C}, A) \rightarrow \text{L}_R(\mathcal{D}, A)$ and it is given by*

$$(g^* \Gamma)_{\mathcal{G}} = \int^{\mathcal{E} \in \mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \text{Hom}_{\mathcal{Y}}(\mathcal{G}, g^* \mathcal{E}) \in \text{Mod } A \text{ for } \Gamma \in \text{L}_R(\mathcal{C}, A), \mathcal{G} \in \mathcal{D}$$

where $\text{Hom}_{\mathcal{Y}}(\mathcal{G}, g^* -)$ is thought of as a functor $\mathcal{C} \rightarrow \text{Mod } R$. If $\mathcal{Y} = \mathcal{X}$ and $g = \text{id}_{\mathcal{X}}$, so that $\mathcal{C} \subseteq \mathcal{D}$ and $(\text{id}_{\mathcal{X}})_*: \text{L}_R(\mathcal{D}, A) \rightarrow \text{L}_R(\mathcal{C}, A)$ is the restriction, then the unit $\Gamma \rightarrow (\text{id}_{\mathcal{X}}^* \Gamma)|_{\mathcal{C}}$ is an isomorphism for $\Gamma \in \text{L}_R(\mathcal{C}, A)$.

Proof. Let $\Omega \in \text{L}_R(\mathcal{D}, A)$. Applying 2.3 with $F = \text{Hom}_{\mathcal{Y}}(\mathcal{G}, g^* -)$ we get a bijection between $\text{Hom}_A((g^* \Gamma)_{\mathcal{G}}, \Omega_{\mathcal{G}})$ and the set of A -linear natural transformations $\Gamma_{\mathcal{E}} \rightarrow \text{Hom}_R(\text{Hom}(\mathcal{G}, g^* \mathcal{E}), \Omega_{\mathcal{G}})$ for $\mathcal{E} \in \mathcal{C}$. A natural transformation $g^* \Gamma \rightarrow \Omega$ corresponds to a collection $\underline{\gamma}$ of A -linear maps $\gamma_{\mathcal{G}, \mathcal{E}}: \Gamma_{\mathcal{E}} \rightarrow \text{Hom}_R(\text{Hom}(\mathcal{G}, g^* \mathcal{E}), \Omega_{\mathcal{G}})$ natural in $\mathcal{E} \in \mathcal{C}$ and such that

$$\gamma_{\mathcal{G}, \mathcal{E}}(x)(\phi \circ u) = \Omega_u(\gamma_{\overline{\mathcal{G}}, \mathcal{E}}(x)(\phi)) \text{ for } x \in \Gamma_{\mathcal{E}}, \overline{\mathcal{G}} \xrightarrow{\phi} g^* \mathcal{E}, \mathcal{G} \xrightarrow{u} \overline{\mathcal{G}} \in \mathcal{D}$$

Set $\mu_{\underline{\gamma}}(x) = \gamma_{g^*\mathcal{E}, \mathcal{E}}(x)(\text{id}_{g^*\mathcal{E}}) \in \Omega_{g^*\mathcal{E}}$ for $\mathcal{E} \in \mathcal{C}, x \in \Gamma_{\mathcal{E}}$. If $\underline{\gamma}$ is a collection as above it follows that $\gamma_{\mathcal{G}, \mathcal{E}}(x)(\psi) = \Omega_{\psi}(\mu_{\underline{\gamma}}(x))$ for $\mathcal{E} \in \mathcal{C}, \mathcal{G} \in \mathcal{D}, x \in \Gamma_{\mathcal{E}}$ and $\mathcal{G} \xrightarrow{\psi} g^*\mathcal{E}$ and that $\mu_{\underline{\gamma}}: \Gamma_{\mathcal{E}} \rightarrow \Omega_{g^*\mathcal{E}} = (g_*\Omega)_{\mathcal{E}}$ is A -linear and natural in \mathcal{E} . Conversely, given a morphism $\mu: \Gamma \rightarrow (g_*\Omega)$ in $L_R(\mathcal{C}, A)$ we can always define a collection $\underline{\gamma}$ as above by setting $\gamma_{\mathcal{G}, \mathcal{E}}(x)(\psi) = \Omega_{\psi}(\mu(x))$. It is easy to check that $\underline{\gamma}$ induces a morphism $g^*\Gamma \rightarrow \Omega$ and that the above constructions yields a bijection $\text{Hom}(g^*\Gamma, \Omega) \simeq \text{Hom}(\Gamma, g_*\Omega)$.

Assume now $\mathcal{Y} = \mathcal{X}$ and $g = \text{id}_{\mathcal{X}}$ and let $\Gamma \in L_R(\mathcal{C}, A)$ and $\bar{\mathcal{E}} \in \mathcal{C}$. Denote by $\alpha: \Gamma \rightarrow (\text{id}_{\mathcal{X}}^*\Gamma)_{|\mathcal{C}}$ the unit morphism. If $p_{\bar{\mathcal{E}}}: \Gamma_{\bar{\mathcal{E}}} \otimes \text{Hom}_{\mathcal{X}}(\bar{\mathcal{E}}, \bar{\mathcal{E}}) \rightarrow (\text{id}_{\mathcal{X}}^*\Gamma)_{\bar{\mathcal{E}}}$ are the structure morphisms as in 2.3, then $\alpha_{\bar{\mathcal{E}}} = p_{\bar{\mathcal{E}}}(- \otimes \text{id}_{\bar{\mathcal{E}}})$. In particular, given $\mathcal{H} \in \text{Mod } A$ and using 2.3, the map $\text{Hom}_A(\alpha_{\bar{\mathcal{E}}}, \mathcal{H}): \text{Hom}_A((\text{id}_{\mathcal{X}}^*\Gamma)_{\bar{\mathcal{E}}}, \mathcal{H}) \rightarrow \text{Hom}_A(\Gamma_{\bar{\mathcal{E}}}, \mathcal{H})$ sends an A -linear natural transformation $\delta: \Gamma_{-} \rightarrow \text{Hom}_R(\text{Hom}_{\mathcal{X}}(\bar{\mathcal{E}}, -), \mathcal{H})$ to $\Gamma_{\bar{\mathcal{E}}} \ni x \mapsto \delta(x)(\text{id}_{\bar{\mathcal{E}}}) \in \mathcal{H}$. Since δ corresponds to an R -linear natural transformation $\text{Hom}_{\mathcal{X}}(\bar{\mathcal{E}}, -) \rightarrow \text{Hom}_A(\Gamma_{-}, \mathcal{H})$, by Yoneda lemma we see that $\text{Hom}_A(\alpha_{\bar{\mathcal{E}}}, \mathcal{H})$ is an isomorphism. \square

The above proposition yields a natural extension of any $\Gamma \in L_R(\mathcal{C}, A)$ to a functor $\Gamma^{ex} \in L_R(\text{QCoh } \mathcal{X}, A)$. By abuse of notation we will denote them by the same symbol Γ . This means that if $\Gamma \in L_R(\mathcal{C}, A)$ and $\mathcal{G} \in \text{QCoh } \mathcal{X}$ then we can evaluate Γ on \mathcal{G} , writing $\Gamma_{\mathcal{G}}$.

Given a map $g: \mathcal{Y} \rightarrow \mathcal{X}$ we will denote by $g^*: L_R(\mathcal{C}, A) \rightarrow L_R(g^*\mathcal{C}, A)$ the left adjoint of the restriction $L_R(g^*\mathcal{C}, A) \rightarrow L_R(\mathcal{C}, A)$. So, given $\Gamma \in L_R(\mathcal{C}, A)$, $g^*\Gamma$ is a functor $g^*\mathcal{C} \rightarrow \text{Mod } A$ but it also defines a functor $\text{QCoh } \mathcal{Y} \rightarrow \text{Mod } A$ denoted, by our convention, by the same symbol. By 2.8 the functor $g^*\Gamma: \text{QCoh } \mathcal{Y} \rightarrow \text{Mod } A$ coincides with the value of the left adjoint of the restriction $L_R(\text{QCoh } \mathcal{Y}, A) \rightarrow L_R(\mathcal{C}, A)$.

Remark 2.9. Given $\Gamma \in L_R(\mathcal{C}, A)$ and $\mathcal{E} \in \mathcal{C}$ we have R -linear morphisms of rings

$$H^0(\mathcal{O}_{\mathcal{X}}) \simeq \text{End}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \rightarrow \text{End}_{\mathcal{X}}(\mathcal{E}) \rightarrow \text{End}_A(\Gamma_{\mathcal{E}})$$

This defines a lifting of Γ to an R -linear functor $\Gamma: \mathcal{C} \rightarrow \text{Mod}(H^0(\mathcal{O}_{\mathcal{X}}) \otimes_R A)$ and an equivalence

$$L_R(\mathcal{C}, A) \rightarrow L_R(\mathcal{C}, H^0(\mathcal{O}_{\mathcal{X}}) \otimes_R A)$$

In particular, if $g: \text{Spec } B \rightarrow \mathcal{X}$ is a map and $\Gamma \in L_R(\mathcal{C}, A)$ then $(g^*\Gamma)_B$ has a $B \otimes_R A$ -module structure. By 2.4 and 2.8 there is a canonical A -linear isomorphism

$$\mathcal{F}_{\Gamma, \mathcal{C}}(B) \simeq (g^*\Gamma)_B$$

and it is easy to see that it is also B -linear.

Proposition 2.10. *Let $g: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of fibered categories. Then there exists an isomorphism $g^*\mathcal{F}_{\Gamma, \mathcal{C}} \simeq \mathcal{F}_{g^*\Gamma, g^*\mathcal{C}}$ natural in $\Gamma \in L_R(\mathcal{C}, A)$, that is a 2-commutative diagram*

$$\begin{array}{ccc} L_R(\mathcal{C}, A) & \xrightarrow{\mathcal{F}_{-, \mathcal{C}}} & \text{QCoh}_A \mathcal{X} \\ \downarrow g^* & & \downarrow g^* \\ L_R(g^*\mathcal{C}, A) & \xrightarrow{\mathcal{F}_{-, g^*\mathcal{C}}} & \text{QCoh}_A \mathcal{Y} \end{array}$$

Proof. Let $\Gamma \in L_R(\mathcal{C}, A)$. Let also $\text{Spec } B \xrightarrow{\xi} \mathcal{Y}$ be a map and $N \in \text{Mod } B \otimes_R A$. Denote by $F: \mathcal{C} \rightarrow \text{Mod } B$ and $G: g^*\mathcal{C} \rightarrow \text{Mod } B$ the functors obtained by taking sections over B . In particular $F = G \circ g^*$. We have isomorphisms

$$\begin{aligned} \text{Hom}_{B \otimes_R A}(\mathcal{F}_{g^*\Gamma, g^*\mathcal{C}}(B), N) &\simeq \text{Hom}_{L_R(g^*\mathcal{C}, A)}(g^*\Gamma, \text{Hom}_B(G, N)) \simeq \text{Hom}_{L_R(\mathcal{C}, A)}(\Gamma, g_* \text{Hom}_B(G, N)) \\ &= \text{Hom}_{L_R(\mathcal{C}, A)}(\Gamma, \text{Hom}_B(F, N)) \simeq \text{Hom}_{B \otimes_R A}((g^*\mathcal{F}_{\Gamma, \mathcal{C}})(B), N) \end{aligned}$$

In particular we get an isomorphism $(g^* \mathcal{F}_{\Gamma, \mathcal{C}})(B) \simeq \mathcal{F}_{g^* \Gamma, g^* \mathcal{C}}(B)$. By a direct check it follows that this isomorphism is natural in Γ and it also extends as an isomorphism of functors $g^* \mathcal{F}_{\Gamma, \mathcal{C}} \simeq \mathcal{F}_{g^* \Gamma, g^* \mathcal{C}}$. \square

Notice that, as soon as $g^*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{QCoh} \mathcal{Y}$ has a right adjoint $g_*: \mathrm{QCoh} \mathcal{Y} \rightarrow \mathrm{QCoh} \mathcal{X}$ (see 1.7), then $g_* \Omega^{\mathcal{G}} \simeq \Omega^{g^* \mathcal{G}}$ for $\mathcal{G} \in \mathrm{QCoh}_A \mathcal{Y}$ and the above proposition follows taking the left adjoints.

Remark 2.11. If $A \rightarrow A'$ is a morphism of R -algebras then we have pull-back functors $\mathrm{L}_R(\mathcal{C}, A) \rightarrow \mathrm{L}_R(\mathcal{C}, A')$ and $\mathrm{QCoh}_A \mathcal{X} \rightarrow \mathrm{QCoh}_{A'} \mathcal{X}$. The first one is obtained considering the tensor product $- \otimes_A A'$, while the second one corresponds to the pullback $\mathrm{QCoh} \mathcal{X}_A \rightarrow \mathrm{QCoh} \mathcal{X}_{A'}$ along the projection $\mathcal{X}_{A'} \rightarrow \mathcal{X}_A$. Alternatively, those functors are left adjoints to the restriction of scalars $\mathrm{L}_R(\mathcal{C}, A') \rightarrow \mathrm{L}_R(\mathcal{C}, A)$ and $\mathrm{QCoh}_{A'} \mathcal{X} \rightarrow \mathrm{QCoh}_A \mathcal{X}$ respectively. It is easy to see that in this way we obtain two fpqc stacks (not in groupoids) $\mathrm{L}_R(\mathcal{C}, -)$ and $\mathrm{QCoh}_- \mathcal{X}$ over the category of affine R -schemes. Notice that the functor $\Omega^*: \mathrm{QCoh}_- \mathcal{X} \rightarrow \mathrm{L}_R(\mathcal{C}, -)$ is not a morphism of stacks because $\mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G}) \otimes_A A' \not\simeq \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G} \otimes_A A')$ in general for $\mathcal{E} \in \mathcal{C}$ and $\mathcal{G} \in \mathrm{QCoh}_A \mathcal{X}$.

Proposition 2.12. *The functor $\mathcal{F}_{*, \mathcal{C}}: \mathrm{L}_R(\mathcal{C}, -) \rightarrow \mathrm{QCoh}_- \mathcal{X}$ is a morphism of stacks.*

Proof. Given a morphism $A \rightarrow A'$ of R -algebras we have a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}_{A'} \mathcal{X} & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{C}, A') \\ \downarrow & & \downarrow \\ \mathrm{QCoh}_A \mathcal{X} & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{C}, A) \end{array}$$

where the vertical arrows are obtained by restricting the scalars from A' to A . Using 2.11 and taking the left adjoint functors of the functors in the diagram we exactly get the 2-commutative diagram expressing the fact that $\mathcal{F}_{*, \mathcal{C}}$ preserves Cartesian arrows. \square

We conclude this section by showing that, when considering sheafification functors $\mathcal{F}_{-, \mathcal{C}}$, we can always reduce problems to the case when \mathcal{C} is an additive category. Moreover in this case the sections of $\mathcal{F}_{-, \mathcal{C}}$ have a nice expression in terms of a direct limit.

Definition 2.13. Given a subcategory \mathcal{D} of $\mathrm{QCoh} \mathcal{X}$ we denote by \mathcal{D}^{\oplus} the subcategory of $\mathrm{QCoh} \mathcal{X}$ whose objects are all finite direct sums of sheaves in \mathcal{D} .

Notice that if \mathcal{D} is small then \mathcal{D}^{\oplus} is small.

Proposition 2.14. *The restriction $\mathrm{L}_R(\mathcal{C}^{\oplus}, A) \rightarrow \mathrm{L}_R(\mathcal{C}, A)$ and its left adjoint are inverses of each other. In particular if $\Gamma \in \mathrm{L}_R(\mathcal{C}^{\oplus}, A)$ then we have a canonical isomorphism $\mathcal{F}_{\Gamma, \mathcal{C}^{\oplus}} \simeq \mathcal{F}_{\Gamma|_{\mathcal{C}}, \mathcal{C}}$.*

Proof. Denote the restriction by α and its left adjoint by β . From 2.8 we know that $\alpha\beta(\Gamma) \simeq \Gamma$ for $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$. Conversely, if $\Gamma \in \mathrm{L}_R(\mathcal{C}^{\oplus}, A)$, we have a canonical morphism $\gamma: \beta\alpha(\Gamma) \rightarrow \Gamma$. If $\mathcal{E} \in \mathcal{C}$ the map $\gamma_{\mathcal{E}}$ is easily seen to be an isomorphism and by additivity this also follows for γ . For the last claim it is enough to note that the composition $\mathrm{QCoh}_A \mathcal{X} \xrightarrow{\Omega^*} \mathrm{L}_R(\mathcal{C}^{\oplus}, A) \rightarrow \mathrm{L}_R(\mathcal{C}, A)$ is exactly $\Omega^*: \mathrm{QCoh}_A \mathcal{X} \rightarrow \mathrm{L}_R(\mathcal{C}, A)$. \square

Remark 2.15. If \mathcal{C} is an R -linear and additive category and $F, G: \mathcal{C} \rightarrow \mathrm{Mod} A$ are R -linear (covariant or contravariant) functors then any natural transformation $\lambda: F \rightarrow G$ of functors of sets is R -linear. Indeed by considering $\mathcal{C}^{\mathrm{op}}$ we can consider only covariant functors. In this case it is easy to show that the maps $\lambda_X: F(X) \rightarrow G(X)$ for $X \in \mathcal{C}$ are R -linear using functoriality on the map $\mathrm{rid}_X: X \rightarrow X$ for $r \in R$ and $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1 + \mathrm{pr}_2: X \oplus X \rightarrow X$, where pr_* are the projections.

Definition 2.16. Let $\text{Spec } B \rightarrow \mathcal{X}$ be a map. We denote by $J_{B,\mathcal{C}}$ the category of pairs (\mathcal{E}, ψ) where $\mathcal{E} \in \mathcal{C}^\oplus$ and $\psi \in \mathcal{E}(B)$. Given $\Gamma \in \text{L}_R(\mathcal{C}, A)$ we have a functor $\Gamma: J_{B,\mathcal{C}} \rightarrow \text{Mod } A$ given by $\Gamma_{\mathcal{E},\psi} = \Gamma_{\mathcal{E}}$.

Proposition 2.17. Let $\text{Spec } B \rightarrow \mathcal{X}$ be a map and $\Gamma \in \text{L}_R(\mathcal{C}, A)$. The category $J_{B,\mathcal{C}}$ is non-empty and for all $\xi, \xi' \in J_{B,\mathcal{C}}$ there exists $\xi'' \in J_{B,\mathcal{C}}$ and maps $\xi'' \rightarrow \xi, \xi'' \rightarrow \xi'$. The A -linear maps $\Gamma_{\mathcal{E},\psi} = \Gamma_{\mathcal{E}} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}^\oplus}(B) \simeq \mathcal{F}_{\Gamma,\mathcal{C}}(B)$, $x \mapsto x_{\mathcal{E},\psi}$ for $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$ (see 2.5) induce an A -linear isomorphism

$$\varinjlim_{(\mathcal{E},\psi) \in J_{B,\mathcal{C}}^{\text{op}}} \Gamma_{\mathcal{E},\psi} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}(B)$$

In particular all elements of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ are of the form $x_{\mathcal{E},\psi}$ for some $\mathcal{E} \in \mathcal{C}^\oplus$, $x \in \Gamma_{\mathcal{E}}$ and $\psi \in \mathcal{E}(B)$. The multiplication by $b \in B$ on the first limit is induced by mapping $\Gamma_{\mathcal{E},\psi}$ to $\Gamma_{\mathcal{E},b\psi}$ using $\text{id}_{\Gamma_{\mathcal{E}}}$ for $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$.

Proof. By 2.14 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Denote by \mathcal{H} and $\alpha: \mathcal{H} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}(B)$ the limit and the map in the statement respectively. The category $J_{B,\mathcal{C}}$ is not empty because $(\mathcal{E}, 0) \in J_{B,\mathcal{C}}$ for all $\mathcal{E} \in \mathcal{C}$ and the map α is well defined because, for all $x \in \Gamma_{\mathcal{E}}$ and for all $(\mathcal{E}, \psi) \xrightarrow{u} (\overline{\mathcal{E}}, u(\psi))$ we have $x_{\overline{\mathcal{E}},u(\psi)} = (\Gamma_u(x))_{\mathcal{E},\psi}$, by definition of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ as coend. Moreover if $(\mathcal{E}_1, \psi_1), (\mathcal{E}_2, \psi_2) \in J_{B,\mathcal{C}}$ then we have maps $\text{pr}_i: (\mathcal{E}_1 \oplus \mathcal{E}_2, \psi_1 \oplus \psi_2) \rightarrow (\mathcal{E}_i, \psi_i)$ for $i = 1, 2$, where pr_i is the projection. In particular any element of \mathcal{H} comes from a map $\Gamma_{\mathcal{E},\psi} \rightarrow \mathcal{H}$, where $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$. Thus all elements of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ are of the form $x_{\mathcal{E},\psi}$ for some $\mathcal{E} \in \mathcal{C}^\oplus$, $x \in \Gamma_{\mathcal{E}}$ and $\psi \in \mathcal{E}(B)$ provided that we prove that α is an isomorphism.

Given an A -module N then $\text{Hom}_A(\mathcal{H}, N)$ is A -linearly isomorphic to the set of natural transformations of sets $\beta_{\mathcal{E}}: \mathcal{E}(B) \rightarrow \text{Hom}_A(\Gamma_{\mathcal{E}}, N)$. Since \mathcal{C} is additive, by 2.15, those transformations are automatically R -linear. Given $b \in B$ there is an A -linear map $\phi_b \in \text{End}_A(\mathcal{H})$ as described in the last sentence in the statement. A direct check shows that, given an A -module N , the map $\text{Hom}_A(\phi_b, N): \text{Hom}_A(\mathcal{H}, N) \rightarrow \text{Hom}_A(\mathcal{H}, N)$ sends an R -linear natural transformation $\beta_{\mathcal{E}}: \mathcal{E}(B) \rightarrow \text{Hom}_A(\Gamma_{\mathcal{E}}, N)$ to $\beta_{\mathcal{E}} \circ \text{bid}_{\mathcal{E}}$. This easily implies that $\phi: B \rightarrow \text{End}_A(\mathcal{H})$ makes \mathcal{H} into a $B \otimes_R A$ -module and that $\alpha: \mathcal{H} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}(B)$ is also B -linear. Since $B \otimes_R A$ -linear maps $\mathcal{H} \rightarrow N$, for $N \in \text{Mod } B \otimes_R A$, corresponds to B -linear natural transformations $\mathcal{E}(B) \rightarrow \text{Hom}_A(\Gamma_{\mathcal{E}}, N)$ and thus to A -linear natural transformations $\Gamma_{\mathcal{E}} \rightarrow \text{Hom}_B(\mathcal{E}(B), N)$ as described above, one can check directly (using 2.3) that $\text{Hom}_{B \otimes_R A}(\alpha, N)$ induces the identity on $\text{Hom}_{\text{L}_R(\mathcal{C}, A)}(\Gamma_{-}, \text{Hom}_B(-(B), N))$. This implies that $\alpha: \mathcal{H} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}(B)$ is an isomorphism. \square

2.2. Sheaffifying R -linear monoidal functors. In this section we show how “ring structures” on a quasi-coherent sheaf over \mathcal{X} correspond to “monoidal” structures on the corresponding Yoneda functor.

We start setting up some definitions:

Definition 2.18. Let \mathcal{C} and \mathcal{D} be R -linear symmetric monoidal categories. A (contravariant) pseudo-monoidal functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is an R -linear (and contravariant) functor together with a natural transformation

$$\iota_{V,W}^\Omega: \Omega_V \otimes \Omega_W \rightarrow \Omega_{V \otimes W} \text{ for } V, W \in \mathcal{C}$$

A (contravariant) pseudo-monoidal functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is

- 1) *symmetric* or commutative if for all $V, W \in \mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V \otimes \Omega_W & \xrightarrow{\iota_{V,W}^\Omega} & \Omega_{V \otimes W} \\ \downarrow & & \downarrow \\ \Omega_W \otimes \Omega_V & \xrightarrow{\iota_{W,V}^\Omega} & \Omega_{W \otimes V} \end{array}$$

where the vertical arrows are the obvious isomorphisms;

- 2) *associative* if for all $V, W, Z \in \mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V \otimes \Omega_W \otimes \Omega_Z & \xrightarrow{\iota_{V,W}^\Omega \otimes \text{id}} & \Omega_{V \otimes W} \otimes \Omega_Z \\ \downarrow \text{id} \otimes \iota_{W,Z}^\Omega & & \downarrow \iota_{V \otimes W, Z}^\Omega \\ \Omega_V \otimes \Omega_{W \otimes Z} & \xrightarrow{\iota_{V,W \otimes Z}^\Omega} & \Omega_{V \otimes W \otimes Z} \end{array}$$

If I and J are the unit objects of \mathcal{C} and \mathcal{D} respectively, a unity for Ω is a morphism $1: J \rightarrow \Omega_I$ such that, for all $V \in \mathcal{C}$, the compositions

$$\Omega_V \otimes J \xrightarrow{\text{id} \otimes 1} \Omega_V \otimes \Omega_I \xrightarrow{\iota_{V,I}^\Omega} \Omega_{V \otimes I} \rightarrow \Omega_V \quad \text{and} \quad J \otimes \Omega_V \xrightarrow{1 \otimes \text{id}} \Omega_I \otimes \Omega_V \xrightarrow{\iota_{I,V}^\Omega} \Omega_{I \otimes V} \rightarrow \Omega_V$$

coincide with the natural isomorphisms $\Omega_V \otimes J \rightarrow \Omega_V$ and $J \otimes \Omega_V \rightarrow \Omega_V$ respectively. A (contravariant) *monoidal* functor $\Omega: \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric and associative pseudo-monoidal (contravariant) functor with a unity 1.

A morphism of pseudo-monoidal functors $(\Omega, \iota^\Omega) \rightarrow (\Gamma, \iota^\Gamma)$, called a monoidal morphism or tranformation, is a natural transformation $\Omega \rightarrow \Gamma$ which commutes with the monoidal structures ι^* . A morphism of monoidal functors is a monoidal transformation preserving the unities.

Definition 2.19. We define the categories:

- $\text{Rings}_A \mathcal{X}$, whose objects are $\mathcal{B} \in \text{QCoh}_A \mathcal{X}$ with an A -linear map $m: \mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{B} \rightarrow \mathcal{B}$, called the multiplication;
- $\text{QAlg}_A \mathcal{X}$, as the (not full) subcategory of $\text{Rings}_A \mathcal{X}$ whose objects are \mathcal{B} with a commutative, associative multiplication with a unity and the arrows are morphisms preserving unities;

We also set $\text{Rings} \mathcal{X} = \text{Rings}_R \mathcal{X}$ and $\text{QAlg} \mathcal{X} = \text{QAlg}_R \mathcal{X}$.

Let \mathcal{D} be a monoidal subcategory of $\text{QCoh} \mathcal{X}$, that is a subcategory such that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$ and for all $\mathcal{E}, \mathcal{E}' \in \mathcal{D}$ we have $\mathcal{E} \otimes \mathcal{E}' \in \mathcal{D}$. We define the category $\text{PML}_R(\mathcal{D}, A)$ (resp. $\text{ML}_R(\mathcal{D}, A)$), whose objects are $\Gamma \in \text{LR}(\mathcal{D}, A)$ with a pseudo-monoidal (resp. monoidal) structure.

Remark 2.20. If $q: \mathcal{X}_A \rightarrow \mathcal{X}$ is the projection, the equivalence $\text{QCoh} \mathcal{X}_A \rightarrow \text{QCoh}_A \mathcal{X}$ extends to an equivalence

$$q_*: \text{QRings} \mathcal{X}_A \rightarrow \text{QRings}_A \mathcal{X}$$

Indeed, if $\mathcal{G} \in \text{QCoh} \mathcal{X}_A$ then $q_*(\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{G}) = q_* \mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}_A}} q_* \mathcal{G}$ (see 1.11). We will use the following notation, which is somehow implicit in the definition of $\text{QAlg}_A \mathcal{X}$: a sheaf $\mathcal{B} \in \text{QRings}_A \mathcal{X}$ with $\mathcal{B} \simeq q_* \mathcal{B}'$ is associative (resp. commutative, has a unity, ...) if \mathcal{B}' has the same property.

If $\mathcal{B} \in \text{QRings}_A \mathcal{X}$ with multiplication m , then the composition $\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{B} \xrightarrow{m} \mathcal{B}$ induces a ring structure on \mathcal{B} as an $\mathcal{O}_{\mathcal{X}}$ -module, i.e. $\mathcal{B} \in \text{QRings} \mathcal{X}$. Moreover $\mathcal{B} \in \text{QRings}_A \mathcal{X}$ is associative (resp. commutative, has a unity) if and only if \mathcal{B} has the same property. If $\mathcal{B} \in \text{QAlg}_A \mathcal{X}$ we can form the relative spectrum $\text{Spec} \mathcal{B}$ over \mathcal{X}_A and also over \mathcal{X} . The final result is the same.

Remark 2.21. For $\mathcal{B} \in \text{Rings}_A \mathcal{X}$ or $\Gamma \in \text{PMon}_R(\mathcal{D}, A)$ having a unity is a property, not an additional datum. Indeed in both cases unities are unique.

Let \mathcal{D} be a monoidal subcategory of $\text{QCoh } \mathcal{X}$. If $\mathcal{B} \in \text{Rings}_A \mathcal{X}$ with multiplication m , we endow $\Omega^{\mathcal{B}} \in \text{L}_R(\mathcal{D}, A)$ with the pseudo monoidal structure

$$\iota^{\mathcal{B}}: \text{Hom}(\mathcal{E}, \mathcal{B}) \otimes_A \text{Hom}(\overline{\mathcal{E}}, \mathcal{B}) \longrightarrow \text{Hom}(\mathcal{E} \otimes \overline{\mathcal{E}}, \mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{B}) \xrightarrow{\text{Hom}(\mathcal{E} \otimes \overline{\mathcal{E}}, m)} \text{Hom}(\mathcal{E} \otimes \overline{\mathcal{E}}, \mathcal{B}), \mathcal{E}, \overline{\mathcal{E}} \in \mathcal{D}$$

that is $\iota_{\mathcal{E}, \overline{\mathcal{E}}}^{\mathcal{B}}(\phi \otimes \psi) = m \circ (\phi \otimes \psi)$. If $1 \in \mathcal{B}$ is a unity then we set $1_{\mathcal{B}} = 1 \in \Omega_{\mathcal{O}_{\mathcal{X}}}^{\mathcal{B}} = H^0(\mathcal{B})$

Proposition 2.22. *The structures defined above yield an extension of the functor $\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{L}_R(\mathcal{D}, A)$ to a functor $\Omega^*: \text{QRings}_A \mathcal{X} \rightarrow \text{PML}_R(\mathcal{D}, A)$. Moreover if $\mathcal{B} \in \text{Rings}_A \mathcal{X}$ is associative (resp. commutative, has a unity $1 \in \mathcal{B}$) then $\Omega^{\mathcal{B}}$ is associative (resp. symmetric, has unity $1_{\mathcal{B}} \in \Omega_{\mathcal{O}_{\mathcal{X}}}^{\mathcal{B}}$). In particular we also get a functor $\Omega^*: \text{QAlg}_A \mathcal{X} \rightarrow \text{ML}_R(\mathcal{D}, A)$.*

Proof. The proof is elementary and it consists of applying the definitions. \square

Let \mathcal{C} be a small monoidal subcategory of $\text{QCoh } \mathcal{X}$. Given $\Gamma \in \text{PMon}_R(\mathcal{C}, A)$ with monoidal structure ι , we define the multiplication $m_{\Gamma}: \mathcal{A}_{\Gamma, \mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}_{\Gamma, \mathcal{C}} \rightarrow \mathcal{A}_{\Gamma, \mathcal{C}}$ on $\mathcal{A}_{\Gamma, \mathcal{C}} = \mathcal{F}_{\Gamma, \mathcal{C}}$ by

$$\mathcal{A}_{\Gamma, \mathcal{C}}(B) \otimes_{B \otimes_R A} \mathcal{A}_{\Gamma, \mathcal{C}}(B) \ni x_{\mathcal{E}, \psi} \otimes \overline{x}_{\overline{\mathcal{E}}, \overline{\psi}} \longrightarrow \iota_{\mathcal{E}, \overline{\mathcal{E}}}(x \otimes \overline{x})_{\mathcal{E} \otimes \overline{\mathcal{E}}, \psi \otimes \overline{\psi}} \in \mathcal{A}_{\Gamma, \mathcal{C}}(B)$$

where $\text{Spec } B \rightarrow \mathcal{X}$ is a map, $\mathcal{E}, \overline{\mathcal{E}} \in \mathcal{C}$, $\psi \in \mathcal{E}(B)$, $\overline{\psi} \in \overline{\mathcal{E}}(B)$, $x \in \Gamma_{\mathcal{E}}$, $\overline{x} \in \Gamma_{\overline{\mathcal{E}}}$ (see 2.5). We continue to denote by $\mathcal{A}_{\Gamma, \mathcal{C}}$ the sheaf $\mathcal{F}_{\Gamma, \mathcal{C}}$ together with the multiplication map just defined. If $1 \in \Gamma_{\mathcal{O}_{\mathcal{X}}}$ is a unity we set $1_{\Gamma} \in \mathcal{A}_{\Gamma, \mathcal{C}}$ the image of 1 under the morphism $\Gamma_{\mathcal{O}_{\mathcal{X}}} \rightarrow \Omega_{\mathcal{O}_{\mathcal{X}}}^{\mathcal{A}_{\Gamma, \mathcal{C}}} = H^0(\mathcal{A}_{\Gamma, \mathcal{C}})$.

Proposition 2.23. *The structures defined above yield an extension of the functor $\mathcal{F}_{*, \mathcal{C}}: \text{L}_R(\mathcal{C}, A) \rightarrow \text{QCoh}_A \mathcal{X}$ to a functor $\mathcal{A}_{*, \mathcal{C}}: \text{PML}_R(\mathcal{C}, A) \rightarrow \text{Rings}_A \mathcal{X}$ which is left adjoint to $\Omega^*: \text{Rings}_A \mathcal{X} \rightarrow \text{PML}_R(\mathcal{C}, A)$. More precisely, if $\mathcal{B} \in \text{Rings}_A \mathcal{X}$ then the morphism $\delta_{\mathcal{B}}: \mathcal{A}_{\Omega^{\mathcal{B}}, \mathcal{C}} \rightarrow \mathcal{B}$ preserves multiplications and unities, while if $\Gamma \in \text{PMon}_R(\mathcal{C}, A)$ then the natural transformation $\gamma_{\Gamma}: \Gamma \rightarrow \Omega^{\mathcal{A}_{\Gamma, \mathcal{C}}}$ is monoidal and preserves unities (see 2.5).*

If $\Gamma \in \text{PMon}_R(\mathcal{C}, A)$ is associative (resp. symmetric, has a unity $1 \in \Gamma_{\mathcal{O}_{\mathcal{X}}}$) then $\mathcal{A}_{\Gamma, \mathcal{C}}$ is associative (resp. commutative, has unity $1_{\Gamma} \in \mathcal{A}_{\Gamma, \mathcal{C}}$). In particular we get a functor $\mathcal{A}_{, \mathcal{C}}: \text{ML}_R(\mathcal{C}, A) \rightarrow \text{QAlg}_A \mathcal{X}$ which is left adjoint to $\Omega^*: \text{QAlg}_A \mathcal{X} \rightarrow \text{ML}_R(\mathcal{C}, A)$.*

Proof. A lot of properties have to be checked. All of them are very easy, because they consist only in the application of the definitions. We therefore leave them to the reader. \square

3. YONEDA EMBEDDINGS.

In this section we address the problem of when the Yoneda functor $\text{QCoh}_A \mathcal{X} \rightarrow \text{L}_R(\mathcal{C}, A)$ is fully faithful and describe its essential image. This will lead us to the notion of generating category and left exactness for functors in $\text{L}_R(\mathcal{C}, A)$.

We fix an R -algebra A and a fibered category \mathcal{X} . We assume that our fibered category \mathcal{X} is pseudo-algebraic, that is we assume there exists a representable map $X \rightarrow \mathcal{X}$ from a scheme which is an fpqc covering. We also fix a small subcategory \mathcal{C} of $\text{QCoh } \mathcal{X}$.

Definition 3.1. Let $\mathcal{D} \subseteq \text{QCoh } \mathcal{X}$ be a subcategory. A sheaf $\mathcal{G} \in \text{QCoh } \mathcal{X}$ is generated by \mathcal{D} if there exists a surjective morphism

$$\bigoplus_{i \in I} \mathcal{E}_i \longrightarrow \mathcal{G}$$

where I is a set and $\mathcal{E}_i \in \mathcal{D}$ for all $i \in I$. A sheaf $\mathcal{G} \in \text{QCoh}_A \mathcal{X}$ is generated by \mathcal{D} if it is so as an object of $\text{QCoh } \mathcal{X}$. Equivalently, a sheaf $\mathcal{G} \in \text{QCoh } \mathcal{X}_A$ is generated by \mathcal{D} if $h_* \mathcal{G} \in \text{QCoh } \mathcal{X}$ is so, where $h: \mathcal{X}_A \rightarrow \mathcal{X}$ is the projection. We define $\text{QCoh}_A^{\mathcal{D}} \mathcal{X}$ as the subcategory of $\text{QCoh}_A \mathcal{X}$

of sheaves \mathcal{G} generated by \mathcal{D} and such that, for all maps $\mathcal{E} \xrightarrow{\psi} \mathcal{G}$ with $\mathcal{E} \in \mathcal{D}^\oplus$, also $\text{Ker } \psi$ is generated by \mathcal{D} .

If \mathcal{D}' is another subcategory of $\text{QCoh } \mathcal{X}$ we will say that \mathcal{D} generates \mathcal{D}' if all quasi-coherent sheaves in \mathcal{D}' are generated by \mathcal{D} .

Remark 3.2. Consider a set of morphisms $\{U_j = \text{Spec } B_j \rightarrow \mathcal{X}\}_{j \in J}$ such that $\sqcup_j U_j \rightarrow \mathcal{X}$ is an atlas. By 1.4 we have the following characterizations. If $\mathcal{G} \in \text{QCoh } \mathcal{X}$ then \mathcal{G} is generated by \mathcal{D} if and only if

$$\forall j \in J, x \in \mathcal{G}(B_j), \exists \mathcal{E} \in \mathcal{D}^\oplus, \phi \in \mathcal{E}(B_j), \mathcal{E} \xrightarrow{u} \mathcal{G} \text{ such that } u(\phi) = x$$

If $\mathcal{E} \in \mathcal{D}^\oplus$ and $\psi: \mathcal{E} \rightarrow \mathcal{G}$ is a map then $\text{Ker } \psi$ is generated by \mathcal{D} if and only

$$\forall j \in J, y \in \mathcal{E}(B_j) \text{ with } \psi(y) = 0, \exists \bar{\mathcal{E}} \in \mathcal{D}^\oplus, \phi \in \bar{\mathcal{E}}(B_j), \bar{\mathcal{E}} \xrightarrow{v} \mathcal{E} \text{ such that } \psi v = 0, v(\phi) = y$$

In particular if $\mathcal{G} \in \text{QCoh}^{\mathcal{D}} \mathcal{X}$, $\mathcal{H} \in \text{QCoh } \mathcal{X}$ is generated by \mathcal{D} and $\mathcal{H} \xrightarrow{\alpha} \mathcal{G}$ is a map then $\text{Ker } \alpha$ is generated by \mathcal{D} .

Proposition 3.3. *If \mathcal{D} is a subcategory of $\text{QCoh } \mathcal{X}$ then the category $\text{QCoh}_A^{\mathcal{D}} \mathcal{X}$ is stable by direct sums. In particular $\mathcal{D} \subseteq \text{QCoh}^{\mathcal{D}} \mathcal{X} \iff \mathcal{D}^\oplus \subseteq \text{QCoh}^{\mathcal{D}} \mathcal{X}$.*

Proof. Let $\mathcal{F}, \mathcal{G} \in \text{QCoh}_A^{\mathcal{D}} \mathcal{X}$. Clearly $\mathcal{F} \oplus \mathcal{G}$ is generated by \mathcal{D} . Now consider a map $\alpha: \mathcal{E} \rightarrow \mathcal{F} \oplus \mathcal{G}$ with $\mathcal{E} \in \mathcal{D}^\oplus$ and write $\alpha = \phi \oplus \psi$. By 3.2 it follows that $\text{Ker } \alpha = \text{Ker}(\phi|_{\text{Ker}(\psi)}: \text{Ker } \psi \rightarrow \mathcal{F})$ is generated by \mathcal{D} because $\text{Ker}(\psi)$ is generated by \mathcal{D} and $\mathcal{F} \in \text{QCoh}_A^{\mathcal{D}} \mathcal{X}$. \square

If $g: U = \text{Spec } B \rightarrow \mathcal{X}$ is a map from a scheme then $(J_{B,C})^{\text{op}}$ (see 2.17) is not a filtered category in general. Thus if $\Gamma \in \text{L}_R(\mathcal{C}, A)$ and $x_{\mathcal{E}, \phi} \in \mathcal{F}_{\Gamma, C}(B)$ it is very difficult to understand when $x_{\mathcal{E}, \phi}$ is zero in $\mathcal{F}_{\Gamma, C}(B)$. Luckily, under some hypothesis this is possible.

Lemma 3.4. *Assume $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$. Then for all flat and representable maps $g: \text{Spec } B \rightarrow \mathcal{X}$ the category $(J_{B,C^\oplus})^{\text{op}}$ is filtered. In this case, given $\Gamma \in \text{L}_R(\mathcal{C}, A)$, an element $x_{\mathcal{E}, \phi} \in \mathcal{F}_{\Gamma, C}(B)$ for $(\mathcal{E}, \phi) \in J_{B,C}$ is zero if and only if there exists $(\bar{\mathcal{E}}, \bar{\phi}) \xrightarrow{u} (\mathcal{E}, \phi)$ in $J_{B,C}$ such that $\Gamma_u(x) = 0$.*

Proof. By 2.14 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. By 2.17 we have to prove that for all maps $\alpha, \beta: (\mathcal{E}, \phi) \rightarrow (\bar{\mathcal{E}}, \bar{\phi})$ in $J_{B,C}$ there exists $u: (\mathcal{E}', \phi') \rightarrow (\mathcal{E}, \phi)$ in $J_{B,C}$ such that $\alpha u = \beta u$. Let $\mathcal{K} = \text{Ker}((\alpha - \beta): \mathcal{E} \rightarrow \bar{\mathcal{E}})$, so that $\phi \in \mathcal{K}(B)$ since g is flat. By assumption \mathcal{K} is generated by \mathcal{C} and, since \mathcal{C} is additive, there exist $\mathcal{E}' \in \mathcal{C}$, a map $u: \mathcal{E}' \rightarrow \mathcal{K}$ and $\phi' \in \mathcal{E}'(B)$ such that $u(\phi') = \phi$. So $(\mathcal{E}', \phi') \xrightarrow{u} (\mathcal{E}, \phi)$ is an arrow in $J_{B,C}$ such that $(\alpha - \beta)u = 0$ as required. The last claim follows from 2.17 and the fact that $J_{B,C}^{\text{op}}$ is filtered. \square

In what follows we work out sufficient (and sometimes necessary) conditions for the surjectivity or injectivity of $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, C} \rightarrow \mathcal{G}$. Recall that $\delta_{\mathcal{G}}(u_{\mathcal{E}, \phi}) = u(\phi)$ for $\mathcal{E} \in \mathcal{C}^\oplus$, $\text{Spec } B \rightarrow \mathcal{X}$, $\phi \in \mathcal{E}(B)$, $u \in \Omega_{\mathcal{E}}^{\mathcal{G}} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G})$ (see 2.6).

Lemma 3.5. *If $\mathcal{G} \in \text{QCoh}_A \mathcal{X}$ the map $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, C} \rightarrow \mathcal{G}$ is surjective if and only if \mathcal{G} is generated by \mathcal{C} .*

Proof. By 2.14 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\{g_i: U_i = \text{Spec } B_i \rightarrow \mathcal{X}\}$ be a set of maps such that $\sqcup_i U_i \rightarrow \mathcal{X}$ is an atlas. By 1.4 $\delta_{\mathcal{G}}$ is surjective if and only if $\delta_{\mathcal{G}, U_i}: \mathcal{F}_{\Omega^{\mathcal{G}}, C}(U_i) \rightarrow \mathcal{G}(U_i)$ is surjective for all $i \in I$. By 2.17 $\text{Im } \delta_{\mathcal{G}, U_i}$ is the set of elements of $\mathcal{G}(U_i)$ of the form $\delta_{\mathcal{G}}(u_{\mathcal{E}, \phi}) = u(\phi)$ for $\mathcal{E} \in \mathcal{C}^\oplus$, $\phi \in \mathcal{E}(U_i)$, $u \in \Omega_{\mathcal{E}}^{\mathcal{G}} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G})$. So the claim follows from 3.2. \square

Lemma 3.6. *Let $\mathcal{G} \in \text{QCoh}_A \mathcal{X}$. If for all maps $\mathcal{E} \xrightarrow{\phi} \mathcal{G}$ with $\mathcal{E} \in \mathcal{C}^\oplus$ the kernel $\text{Ker } \phi$ is generated by \mathcal{C} then the map $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, C} \rightarrow \mathcal{G}$ is injective. The converse holds if $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$.*

Proof. By 2.14 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\{g_i: U_i = \text{Spec } B_i \rightarrow \mathcal{X}\}$ be a set of maps such that $\sqcup_i U_i \rightarrow \mathcal{X}$ is an atlas. By 1.4 $\delta_{\mathcal{G}}$ is injective if and only if $\delta_{\mathcal{G}, U_i}: \mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}}(U_i) \rightarrow \mathcal{G}(U_i)$ is injective for all $i \in I$. We start proving that $\delta_{\mathcal{G}}$ is injective if the hypothesis in the statement is fulfilled. So let $z \in \text{Ker } \delta_{\mathcal{G}, U_i}$. By 2.17 there exists $\mathcal{E} \in \mathcal{C}$, $\phi \in \mathcal{E}(U_i)$ and $u: \mathcal{E} \rightarrow \mathcal{G}$ such that $z = u_{\mathcal{E}, \phi}$ and $\delta_{\mathcal{G}}(u_{\mathcal{E}, \phi}) = u(\phi) = 0$. Set $\mathcal{K} = \text{Ker } u$. Since $\phi \in \mathcal{K}(U_i)$ and, by hypothesis, \mathcal{K} is generated by \mathcal{C} there exist $\bar{\mathcal{E}} \in \mathcal{C}$, $\bar{\phi} \in \bar{\mathcal{E}}(U_i)$ and a map $v: \bar{\mathcal{E}} \rightarrow \mathcal{K}$ such that $v(\bar{\phi}) = \phi$. If we denote by v also the composition $\bar{\mathcal{E}} \rightarrow \mathcal{K} \rightarrow \mathcal{E}$ we have

$$u_{\mathcal{E}, \phi} = (\Omega_v^{\mathcal{G}}(u))_{\bar{\mathcal{E}}, \bar{\phi}} = (uv)_{\bar{\mathcal{E}}, \bar{\phi}} = 0_{\bar{\mathcal{E}}, \bar{\phi}} = 0 \text{ in } \mathcal{F}_{\Gamma, \mathcal{C}}(U_i)$$

Assume now that $\delta_{\mathcal{G}}$ is injective and $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$ and let $u: \mathcal{E} \rightarrow \mathcal{G}$ be a map with $\mathcal{E} \in \mathcal{C}$. We have to prove that $\mathcal{K} = \text{Ker } u$ is generated by \mathcal{C} . If $\phi \in \mathcal{K}(U_i) \subseteq \mathcal{E}(U_i)$, then $u(\phi) = \delta_{\mathcal{G}}(u_{\mathcal{E}, \phi}) = 0$. So $u_{\mathcal{E}, \phi} = 0$ and the conclusion follows from 3.2 and 3.4. \square

In general we can still conclude that:

Proposition 3.7. *If $\mathcal{E} \in \mathcal{C}^\oplus$ then the map $\delta_{\mathcal{E}}: \mathcal{F}_{\Omega^{\mathcal{E}}, \mathcal{C}} \rightarrow \mathcal{E}$ is an isomorphism.*

Proof. By 2.14 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\mathcal{H} \in \text{QCoh} \mathcal{X}$. The map

$$\text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{H}) \xrightarrow{\Omega^*} \text{Hom}_{\text{L}_R(\mathcal{C}, R)}(\Omega^{\mathcal{E}}, \Omega^{\mathcal{H}}) \simeq \text{Hom}_{\mathcal{X}}(\mathcal{F}_{\Omega^{\mathcal{E}}, \mathcal{C}}, \mathcal{H})$$

maps $\text{id}_{\mathcal{E}}$ to $\delta_{\mathcal{E}}$ and thus is induced by $\delta_{\mathcal{E}}$. By the enriched Yoneda's lemma or a direct check we see that the above map and therefore $\delta_{\mathcal{E}}$ are isomorphisms. \square

Theorem 3.8. *Let \mathcal{D}_A be the subcategory of $\text{QCoh}_A \mathcal{X}$ of sheaves \mathcal{G} such that $\delta_{\mathcal{G}}: \mathcal{F}_{\Omega^{\mathcal{G}}, \mathcal{C}} \rightarrow \mathcal{G}$ is an isomorphism. Then \mathcal{D}_A is an additive category containing $\text{QCoh}_A^{\mathcal{C}} \mathcal{X}$, $\mathcal{C}^\oplus \subseteq \mathcal{D}_A$ and the functor*

$$\Omega^*: \mathcal{D}_A \rightarrow \text{L}_R(\mathcal{C}, A)$$

is fully faithful. Moreover if $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$ then $\mathcal{D}_A = \text{QCoh}_A^{\mathcal{C}} \mathcal{X}$.

Proof. The category \mathcal{D}_A is additive because Ω^* and $\mathcal{F}_{*, \mathcal{C}}$ are additive. All the other claims follows from 3.5, 3.6, 3.7 and the fact that $\delta_{\mathcal{G}}$ is the counit of an adjunction. \square

Now we want to address the problem of what is the essential image of the Yoneda functor $\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{L}_R(\mathcal{C}, A)$. We will see that if $\mathcal{F} \in \text{QCoh}_A \mathcal{X}$ the associated Yoneda functor $\Omega^{\mathcal{F}}$ is always “left exact” and we will give sufficient conditions assuring that “left exact” functors in $\text{L}_R(\mathcal{C}, A)$ are Yoneda functors associated with some quasi-coherent sheaf on \mathcal{X} . Since \mathcal{C} is not abelian, we introduce an ad hoc notion of left exactness.

Definition 3.9. Let \mathcal{D} be a subcategory of $\text{QCoh} \mathcal{X}$. A *test sequence* for \mathcal{D} is an exact sequence

$$(3.1) \quad \bigoplus_{k \in K} \mathcal{E}_k \rightarrow \bigoplus_{j \in J} \mathcal{E}_j \rightarrow \mathcal{E} \rightarrow 0 \text{ with } \mathcal{E}, \mathcal{E}_j, \mathcal{E}_k \in \mathcal{D} \text{ for all } j \in J, k \in K$$

in $\text{QCoh} \mathcal{X}$ given by maps $u_j: \mathcal{E}_j \rightarrow \mathcal{E}$, $u_{kj}: \mathcal{E}_k \rightarrow \mathcal{E}_j$ such that for all $k \in K$ the set $\{j \in J \mid u_{kj} \neq 0\}$ is finite. We will also say that it is a test sequence for $\mathcal{E} \in \mathcal{D}$. A finite test sequence for $\mathcal{E} \in \mathcal{D}$ is an exact sequence

$$\mathcal{E}'' \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow 0 \text{ with } \mathcal{E}', \mathcal{E}'' \in \mathcal{D}^\oplus$$

Given $\Gamma \in \text{L}_R(\mathcal{D}, A)$ we say that Γ is exact on the test sequence (3.1) if the sequence

$$(3.2) \quad \begin{array}{ccccccc} & & & (x_j)_j & \longmapsto & (\sum_j \Gamma_{u_{kj}}(x_j))_k & \\ & & & & & & \\ 0 & \longrightarrow & \Gamma_{\mathcal{E}} & \longrightarrow & \prod_{j \in J} \Gamma_{\mathcal{E}_j} & \longrightarrow & \prod_{k \in K} \Gamma_{\mathcal{E}_k} \\ & & & x & \longmapsto & (\Gamma_{u_j}(x))_j & \end{array}$$

is exact. We say that Γ is left exact if it is left exact on all short exact sequences in \mathcal{D} . We define $\text{Lex}_R(\mathcal{D}, A)$ as the subcategory of $\text{L}_R(\mathcal{D}, A)$ of functors exact on all test sequences in \mathcal{D} .

Remark 3.10. Notice that the sequence (3.2) is a complex because Γ is R -linear. Moreover we should warn the reader that the sequence (3.2) is not obtained applying Γ on the test sequence, unless J and K are finite, even if Γ is defined (or extended) to the whole $\text{QCoh } \mathcal{X}$. The problem is that Γ does not necessarily transforms infinite direct sums in products.

Proposition 3.11. *Let $\mathcal{D} \subseteq \text{QCoh } \mathcal{X}$ be a subcategory. If $\mathcal{F} \in \text{QCoh}_A \mathcal{X}$ then $\Omega^{\mathcal{F}} \in \text{Lex}_R(\mathcal{D}, A)$.*

Proof. It is enough to apply $\text{Hom}_{\mathcal{X}}(-, \mathcal{F})$ to the test sequence (3.1) and observe that

$$\text{Hom}_{\mathcal{X}}\left(\bigoplus_i \mathcal{E}_i, \mathcal{F}\right) \simeq \prod_i \text{Hom}(\mathcal{E}_i, \mathcal{F}) \simeq \prod_i \Omega_{\mathcal{E}_i}^{\mathcal{F}}$$

□

Proposition 3.12. *The functor $\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{L}_R(\mathcal{C}, A)$ is left exact. If $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$ then $\mathcal{F}_{*,\mathcal{C}}: \text{L}_R(\mathcal{C}, A) \rightarrow \text{QCoh}_A \mathcal{X}$ is exact.*

Proof. For the first claim it is enough to use that $\text{Hom}_{\mathcal{X}}(\mathcal{E}, -)$ is left exact. For the last part of the statement consider a set of maps $\{U_i = \text{Spec } B_i \xrightarrow{g_i} \mathcal{X}\}_{i \in I}$ such that $\sqcup_i U_i \rightarrow \mathcal{X}$ is an atlas. Let also $\Gamma' \rightarrow \Gamma \rightarrow \Gamma''$ be an exact sequence in $\text{L}_R(\mathcal{C}, A)$. By 2.17 the sequence $\mathcal{F}_{\Gamma',\mathcal{C}}(B_i) \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}(B_i) \rightarrow \mathcal{F}_{\Gamma'',\mathcal{C}}(B_i)$ are exact for all $i \in I$ because limit of exact sequences $\Gamma'_{\mathcal{E},\phi} \rightarrow \Gamma_{\mathcal{E},\phi} \rightarrow \Gamma''_{\mathcal{E},\phi}$ over the category $(J_{B_i,\mathcal{C}})^{\text{op}}$, which is filtered thanks to 3.4. Applying 1.4 we get the result. □

Recall that if $\Gamma \in \text{L}_R(\mathcal{C}, A)$ then $\gamma_{\Gamma,\mathcal{E}}: \Gamma_{\mathcal{E}} \rightarrow \Omega_{\mathcal{E}}^{\mathcal{F}_{\Gamma,\mathcal{C}}} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}_{\Gamma,\mathcal{C}})$ is given by $\gamma_{\Gamma,\mathcal{E}}(x)(\phi) = x_{\mathcal{E},\phi}$ for $\mathcal{E} \in \mathcal{C}$, $x \in \Gamma_{\mathcal{E}}$, $\text{Spec } B \rightarrow \mathcal{X}$ and $\phi \in \mathcal{E}(B)$ (see 2.6).

Lemma 3.13. *Assume $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$. If $\Gamma \in \text{Lex}_R(\mathcal{C}, A)$ and the map $\Omega^* \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_{\Gamma}): \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}} \rightarrow \Omega^{\mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}}}$ is an isomorphism then the natural transformation $\gamma_{\Gamma}: \Gamma \rightarrow \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is an isomorphism.*

Proof. Let $\{U_i = \text{Spec } B_i \xrightarrow{g_i} \mathcal{X}\}_{i \in I}$ be a set of maps such that $\sqcup_i U_i \rightarrow \mathcal{X}$ is an atlas and let $\Psi \in \text{L}_R(\mathcal{C}, A)$ and $x \in \text{Ker } \gamma_{\Psi,\mathcal{E}}$ for some $\mathcal{E} \in \mathcal{C}$. We are going to prove that there exists a surjective map $\mu = \oplus_j \mu_j: \oplus_{j \in J} \mathcal{E}_j \rightarrow \mathcal{E}$ with $\mathcal{E}_j, \mathcal{E} \in \mathcal{C}$ such that $\Psi_{u_j}(x) = 0$ for all j .

If $\phi \in \mathcal{E}(U_i)$, by 3.4 and the fact that $\gamma_{\Psi,\mathcal{E}}(x)(\phi) = x_{\mathcal{E},\phi}$ is zero in $\mathcal{F}_{\Psi,\mathcal{C}}(U_i)$, there exists $(\mathcal{E}_{\phi}, y_{\phi}) \in J_{B_i,\mathcal{C}}$ and a map $(\mathcal{E}_{\phi}, y_{\phi}) \xrightarrow{u_{\phi}} (\mathcal{E}, \phi)$ such that $\Psi_{u_{\phi}}(x) = 0$. Consider the induced map

$$\bigoplus_{i \in I} \bigoplus_{\phi \in \mathcal{E}(U_i)} \mathcal{E}_{\phi} \rightarrow \mathcal{E}$$

which is surjective by 1.4. Writing all the $\mathcal{E}_{\phi} \in \mathcal{C}^{\oplus}$ as sums of sheaves in \mathcal{C} we get the desired surjective map.

We return now to the proof of the statement. Given $x \in \text{Ker } \gamma_{\Gamma,\mathcal{E}}$ and considering a surjection μ as above for $\Psi = \Gamma$, we can conclude that $x = 0$ by 3.24. This means that the natural transformation $\gamma_{\Gamma}: \Gamma \rightarrow \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is injective. Set now $\Pi = \text{Coker } \gamma_{\Gamma,\mathcal{C}}$. By 3.12 we have an exact sequence

$$0 \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}} \xrightarrow{\mathcal{F}_{*,\mathcal{C}}(\gamma_{\Gamma})} \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}} \rightarrow \mathcal{F}_{\Pi,\mathcal{C}} \rightarrow 0$$

This is a split sequence because the composition of $\mathcal{F}_{*,\mathcal{C}}(\gamma_{\Gamma}): \mathcal{F}_{\Gamma,\mathcal{C}} \rightarrow \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}}$ and $\delta_{\mathcal{F}_{\Gamma,\mathcal{C}}}: \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}} \rightarrow \mathcal{F}_{\Gamma,\mathcal{C}}$ is the identity. So Ω^* maintains the exactness of the above sequence and therefore

$$\Omega^{\mathcal{F}_{\Pi,\mathcal{C}}} = \text{Coker}(\Omega^* \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_{\Gamma})) = 0$$

We want to prove that $\Pi = 0$. Let $x \in \Pi_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{C}$. Since $\Omega^{\mathcal{F}_{\Pi,\mathcal{C}}} = 0$ we have $x \in \text{Ker } \gamma_{\Pi,\mathcal{E}}$. Consider a surjection $\mu = \oplus_j \mu_j$ constructed as above starting from $x \in \Pi_{\mathcal{E}}$ and $\Psi = \Pi$. By

3.2 μ can be extended to a test sequence $\oplus_k \mathcal{E}_k \rightarrow \oplus_j \mathcal{E}_j \xrightarrow{\mu} \mathcal{E}$ because $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$. Since $\Omega^{\mathcal{F}_{\Gamma, \mathcal{C}}} \in \mathrm{Lex}_R(\mathcal{C}, A)$ by 3.11 we get a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Gamma_{\mathcal{E}} & \longrightarrow & \Omega_{\mathcal{E}}^{\mathcal{F}_{\Gamma, \mathcal{C}}} & \longrightarrow & \Pi_{\mathcal{E}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \beta \\
 0 & \longrightarrow & \prod_{j \in J} \Gamma_{\mathcal{E}_j} & \longrightarrow & \prod_{j \in J} \Omega_{\mathcal{E}_j}^{\mathcal{F}_{\Gamma, \mathcal{C}}} & \longrightarrow & \prod_{j \in J} \Pi_{\mathcal{E}_j} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \prod_{k \in K} \Gamma_{\mathcal{E}_k} & \longrightarrow & \prod_{k \in K} \Omega_{\mathcal{E}_k}^{\mathcal{F}_{\Gamma, \mathcal{C}}} & &
 \end{array}$$

in which all the rows and the first two columns are exact. By diagram chasing it is easy to conclude that β is injective. Since by construction $\beta(x) = 0$ we can conclude that $x = 0$. \square

Definition 3.14. We define $\mathrm{Lex}_R^{\mathcal{C}}(A)$ as the subcategory of $\mathrm{Lex}_R(\mathcal{C}, A)$ of functors Γ such that $\mathcal{F}_{\Gamma, \mathcal{C}} \in \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X}$.

Theorem 3.15. Assume $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$. Then the functors

$$\Omega^*: \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X} \longrightarrow \mathrm{Lex}_R^{\mathcal{C}}(A) \text{ and } \mathcal{F}_{*, \mathcal{C}}: \mathrm{Lex}_R^{\mathcal{C}}(A) \longrightarrow \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X}$$

are quasi-inverses of each other.

Proof. Let $\Gamma \in \mathrm{Lex}_R(\mathcal{C}, A)$ be such that $\mathcal{F}_{\Gamma, \mathcal{C}} \in \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X}$. The composition

$$\mathcal{F}_{\Gamma, \mathcal{C}} \xrightarrow{\mathcal{F}_{*, \mathcal{C}}(\gamma_{\Gamma})} \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma, \mathcal{C}}}, \mathcal{C}} \xrightarrow{\delta_{\mathcal{F}_{\Gamma, \mathcal{C}}}} \mathcal{F}_{\Gamma, \mathcal{C}}$$

is the identity and $\delta_{\mathcal{F}_{\Gamma, \mathcal{C}}}$ is an isomorphism since $\mathcal{F}_{\Gamma, \mathcal{C}} \in \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X}$ by 3.8. Thus $\mathcal{F}_{*, \mathcal{C}}(\gamma_{\Gamma})$ and therefore $\Omega^* \circ \mathcal{F}_{*, \mathcal{C}}(\gamma_{\Gamma})$ are isomorphisms. By 3.13 we can conclude that if $\Gamma \in \mathrm{Lex}_R^{\mathcal{C}}(A)$ then $\gamma_{\Gamma}: \Gamma \rightarrow \Omega^{\mathcal{F}_{\Gamma, \mathcal{C}}}$ is an isomorphism. The result then follows from 3.8 and 3.11. \square

The following result allow us to extend results from small subcategories of $\mathrm{QCoh} \mathcal{X}$ to any subcategory.

Proposition 3.16. The category $\mathrm{QCoh} \mathcal{X}$ is generated by a small subcategory. Equivalently $\mathrm{QCoh} \mathcal{X}$ has a generator, that is there exists $\mathcal{E} \in \mathrm{QCoh} \mathcal{X}$ such that $\{\mathcal{E}\}$ generates $\mathrm{QCoh} \mathcal{X}$.

Proof. Follows from 1.4 and [SP014, Tag 0780]. \square

Remark 3.17. If $\mathcal{D} \subseteq \mathrm{QCoh} \mathcal{X}$ generates $\mathrm{QCoh} \mathcal{X}$ there always exists a small subcategory $\overline{\mathcal{D}} \subseteq \mathcal{D}$ that generates $\mathrm{QCoh} \mathcal{X}$. Indeed if \mathcal{E} is a generator of $\mathrm{QCoh} \mathcal{X}$ it is enough to take a subset of sheaves in \mathcal{D} that generates \mathcal{E} .

Theorem 3.18. Let $\mathcal{D} \subseteq \mathrm{QCoh} \mathcal{X}$ be a subcategory that generates $\mathrm{QCoh} \mathcal{X}$. Then the functor

$$\Omega^*: \mathrm{QCoh}_A \mathcal{X} \longrightarrow \mathrm{Lex}_R(\mathcal{D}, A)$$

is an equivalence of categories and, if \mathcal{D} is small, $\mathcal{F}_{*, \mathcal{D}}: \mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{QCoh}_A \mathcal{X}$ is a quasi-inverse. In particular if $\overline{\mathcal{D}} \subseteq \mathcal{D}$ is a subcategory that generates $\mathrm{QCoh} \mathcal{X}$ the restriction functor $\mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{Lex}_R(\overline{\mathcal{D}}, A)$ is an equivalence.

Proof. If \mathcal{D} is small we have $\mathrm{QCoh}_A \mathcal{X} = \mathrm{QCoh}_A^{\mathcal{D}} \mathcal{X}$, $\mathrm{Lex}_R(\mathcal{D}, A) = \mathrm{Lex}_R^{\mathcal{D}}(A)$ and everything follows from 3.15. In particular the restriction $\mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{Lex}_R(\overline{\mathcal{D}}, A)$ is an equivalence if $\overline{\mathcal{D}} \subseteq \mathcal{D}$, they are small and they generate $\mathrm{QCoh} \mathcal{X}$. Assume now that \mathcal{D} is not necessarily small and consider a small subcategory $\mathcal{C} \subseteq \mathcal{D}$ that generates $\mathrm{QCoh} \mathcal{X}$, which exists thanks to 3.17. The proof of the statement is complete if we prove that the restriction functor $\mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{Lex}_R(\mathcal{C}, A)$ is an equivalence. Given a set $I \subseteq \mathcal{D}$ we set $\mathcal{C}_I = \mathcal{C} \cup I \subseteq \mathcal{D}$. For all sets I , quasi-coherent sheaves are generated by \mathcal{C}_I . Note that we have the restriction functor $-|_{\mathcal{C}_I}: \mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{Lex}_R(\mathcal{C}_I, A)$ and the composition $\mathrm{QCoh}_A \mathcal{X} \xrightarrow{\Omega^*} \mathrm{Lex}_R(\mathcal{D}, A) \xrightarrow{-|_{\mathcal{C}_I}} \mathrm{Lex}_R(\mathcal{C}_I, A)$ is an equivalence for all I . In particular $-|_{\mathcal{C}}: \mathrm{Lex}_R(\mathcal{D}, A) \rightarrow \mathrm{Lex}_R(\mathcal{C}, A)$ is essentially surjective. We will conclude by proving that it is fully faithful. Let $\Gamma, \Gamma' \in \mathrm{Lex}_R(\mathcal{D}, A)$. If $\overline{I} \subseteq I$ the restriction functor $\mathrm{Lex}_R(\mathcal{C}_I, A) \rightarrow \mathrm{Lex}_R(\mathcal{C}_{\overline{I}}, A)$ is an equivalence of categories. In particular the map $\mathrm{Hom}(\Gamma|_{\mathcal{C}_I}, \Gamma'|_{\mathcal{C}_I}) \rightarrow \mathrm{Hom}(\Gamma|_{\mathcal{C}_{\overline{I}}}, \Gamma'|_{\mathcal{C}_{\overline{I}}})$ is bijective. Using this, it is elementary to prove that also the map $\mathrm{Hom}(\Gamma, \Gamma') \rightarrow \mathrm{Hom}(\Gamma|_{\mathcal{C}}, \Gamma'|_{\mathcal{C}})$ is bijective. \square

As a corollary we recover Gabriel-Popescu's theorem for the category $\mathrm{QCoh} \mathcal{X}$.

Theorem 3.19. *[Gabriel-Popescu's theorem] If \mathcal{E} is a generator of $\mathrm{QCoh} \mathcal{X}$ then the functor*

$$\mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, -): \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{Mod}_{\mathrm{right}}(\mathrm{End}_{\mathcal{X}}(\mathcal{E}))$$

is fully faithful and has an exact left adjoint.

Proof. It follows from 3.18 and 3.12 with $\mathcal{D} = \mathcal{C} = \{\mathcal{E}\}$. \square

Before showing other applications of 3.18 we want to present a cohomological interpretation of the functors in $\mathrm{Lex}_R(\mathcal{C}, A)$, which will allow us to show that it is often enough to consider just finite test sequences instead of arbitrary test sequences.

Remark 3.20. In an abelian category \mathcal{A} , given $X, Y \in \mathcal{A}$ we can always define the abelian group $\mathrm{Ext}^1(X, Y)$ as the group of extensions (regardless if \mathcal{A} has enough injectives) and it has the usual nice properties on short exact sequences. In order to avoid set-theoretic problems one should require that \mathcal{A} is locally small and that, given $X, Y \in \mathcal{A}$, $\mathrm{Ext}^1(X, Y)$ is a set. This is the case for $\mathcal{A} = \mathrm{L}_R(\mathcal{C}, R)$, for instance by looking at the cardinalities of the $\Gamma_{\mathcal{E}}$ for $\Gamma \in \mathrm{L}_R(\mathcal{C}, R)$ and $\mathcal{E} \in \mathcal{C}$.

Definition 3.21. Given a surjective map $\mu = \oplus_j \mu_j: \oplus_j \mathcal{E}_j \rightarrow \mathcal{E}$ with $\mathcal{E}, \mathcal{E}_j \in \mathcal{C}$ we set $\Omega^{\mu} = \oplus_j \Omega^{\mu_j}: \oplus_j \Omega^{\mathcal{E}_j} \rightarrow \Omega^{\mathcal{E}}$. A functor $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ is *cohomologically left exact* on μ if

$$(3.3) \quad \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\Omega^{\mathcal{E}} / \mathrm{Im}(\Omega^{\mu}), \Gamma) = \mathrm{Ext}_{\mathrm{L}_R(\mathcal{C}, R)}^1(\Omega^{\mathcal{E}} / \mathrm{Im}(\Omega^{\mu}), \Gamma) = 0$$

It is cohomologically left exact if it is so on all surjections μ as above.

We setup some notation. We denote by $\Phi_{\mathcal{C}}(\mathcal{E})$ for $\mathcal{E} \in \mathcal{C}$ the set of subfunctor of $\Omega^{\mathcal{E}}$ of the form $\mathrm{Im}(\Omega^{\mu})$ for some surjective map $\mu: \oplus_j \mathcal{E}_j \rightarrow \mathcal{E}$ with $\mathcal{E}_j \in \mathcal{C}$ and by $\Phi_{\mathcal{C}}$ the disjoint union of all the $\Phi_{\mathcal{C}}(\mathcal{E})$. Given $\Delta = \mathrm{Im}(\Omega^{\mu}) \in \Phi_{\mathcal{C}}$ and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ we will say that Γ is cohomologically left exact on Δ if it is cohomologically left exact on μ . Notice that, using Yoneda's lemma, we have an A -linear isomorphism

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\oplus_j \Omega^{\mathcal{E}_j}, \Gamma) \simeq \prod_j \Gamma_{\mathcal{E}_j} \text{ for all } \mathcal{E}_j \in \mathcal{C}, \Gamma \in \mathrm{L}_R(\mathcal{C}, A)$$

In particular $\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\Omega^{\mathcal{E}}, -)$, which is the evaluation in \mathcal{E} , is exact, which implies that $\mathrm{Ext}_{\mathrm{L}_R(\mathcal{C}, R)}^1(\Omega^{\mathcal{E}}, -) = 0$.

A map $\oplus_j \mathcal{E}_j \rightarrow \oplus_k \mathcal{E}_k$ is called *locally finite* if, for all j , the restriction $\mathcal{E}_j \rightarrow \oplus_k \mathcal{E}_k$ factors through a finite sub-sum. Composition of locally finite maps is locally finite and, using Yoneda's lemma, we obtain a functorial map

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\oplus_j \Omega^{\mathcal{E}_j}, \oplus_k \Omega^{\mathcal{E}_k}) \rightarrow \mathrm{Hom}(\oplus_j \mathcal{E}_j, \oplus_k \mathcal{E}_k)$$

which is an isomorphism onto the set of locally finite maps.

Let $\mu: \oplus_j \mathcal{E}_j \rightarrow \mathcal{E}$ be a surjective map and set $\Delta = \mathrm{Im}(\Omega^\mu)$. Notice that $\Delta_{\mathcal{E}'}$ is the set of maps $\mathcal{E}' \rightarrow \mathcal{E}$ which factors through μ via a locally finite map $\mathcal{E}' \rightarrow \oplus_j \mathcal{E}_j$. Given a map $u: \overline{\mathcal{E}} \rightarrow \mathcal{E}$ in \mathcal{C} we set $u^{-1}(\Delta) = \Delta \times_{\Omega^\mathcal{E}} \Omega^{\overline{\mathcal{E}}} \subseteq \Omega^{\overline{\mathcal{E}}}$: $u^{-1}(\Delta)_{\mathcal{E}'}$ is the set of maps $\mathcal{E}' \rightarrow \overline{\mathcal{E}}$ such that $\mathcal{E}' \rightarrow \overline{\mathcal{E}} \rightarrow \mathcal{E}$ factors through μ via a locally finite map $\mathcal{E}' \rightarrow \oplus_j \mathcal{E}_j$. Notice that if $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then $u^{-1}(\Delta) \in \Phi_{\mathcal{C}}(\overline{\mathcal{E}})$. Indeed, if \mathcal{H} is the kernel of $\mu \oplus (-u): (\oplus_j \mathcal{E}_j) \oplus \overline{\mathcal{E}} \rightarrow \mathcal{E}$ then all elements of \mathcal{H} on some object $\mathrm{Spec} B \rightarrow \mathcal{X}$ lies in a finite subsum of $(\oplus_j \mathcal{E}_j) \oplus \overline{\mathcal{E}}$. By 3.2 \mathcal{H} is the image of a locally finite map $\oplus_q \mathcal{E}_q \rightarrow (\oplus_j \mathcal{E}_j) \oplus \overline{\mathcal{E}}$ and, since $\mathcal{H} \rightarrow \overline{\mathcal{E}}$ is surjective, the induced map $\mu': \oplus_{\mathcal{E}' \oplus_{\phi \in u^{-1}(\Delta)_{\mathcal{E}'}} \mathcal{E}'} \rightarrow \overline{\mathcal{E}}$ is surjective and clearly $u^{-1}(\Delta) = \mathrm{Im}(\Omega^{\mu'})$.

Lemma 3.22. *Let $\mu: \oplus_j \mathcal{E}_j \rightarrow \mathcal{E}$ be a surjective map with $\mathcal{E}_j, \mathcal{E} \in \mathcal{C}$ and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$. Then there is an exact sequence of A -modules*

$$0 \rightarrow \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\Omega^\mathcal{E} / \mathrm{Im}(\Omega^\mu), \Gamma) \rightarrow \Gamma_\mathcal{E} \rightarrow \prod_j \Gamma_{\mathcal{E}_j}$$

If $\mathcal{T}: \oplus_k \mathcal{E}_k \rightarrow \oplus_j \mathcal{E}_j \xrightarrow{\mu} \mathcal{E} \rightarrow 0$ is a test sequence and Γ is exact on \mathcal{T} then Γ is cohomologically left exact on μ . The converse holds if the map

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\mathrm{Ker}(\Omega^\mu), \Gamma) \rightarrow \prod_k \Gamma_{\mathcal{E}_k}$$

obtained applying $\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(-, \Gamma)$ to the map $\oplus_k \Omega^{\mathcal{E}_k} \rightarrow \mathrm{Ker}(\Omega^\mu)$ is injective.

Proof. Set $\Delta = \mathrm{Im}(\Omega^\mu)$, $K = \mathrm{Ker}(\Omega^\mu)$. Consider the diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(\Omega^\mathcal{E} / \Delta, \Gamma) & \hookrightarrow & \Gamma_\mathcal{E} & \xrightarrow{\alpha} & \mathrm{Hom}(\Delta, \Gamma) & \rightarrow & \mathrm{Ext}^1(\Omega^\mathcal{E} / \Delta, \Gamma) \rightarrow \mathrm{Ext}^1(\Omega^\mathcal{E}, \Gamma) \\ & & & \searrow & \downarrow & & \\ & & & & \prod_j \Gamma_{\mathcal{E}_j} & & \\ & & & & \downarrow & \searrow \beta & \\ & & & & \mathrm{Hom}(K, \Gamma) & \xrightarrow{\lambda} & \prod_k \Gamma_{\mathcal{E}_k} \end{array}$$

The convention here is that β and λ are defined only when a test sequence \mathcal{T} as in the statement exists and we will not use them for the first statement. All the other maps are obtained splitting $\Omega^{\mathcal{E}_j} \rightarrow \Omega^\mathcal{E} \rightarrow 0$ into two exact sequences and applying $\mathrm{Hom}(-, \Gamma)$, so that the first line and the central column are exact. The map α obtained as composition is the map defined in the first sequence in the statement. In particular the first claim follows. So let's focus on the second one. The map λ is the second map in the statement while the map β together with α are the maps defining the sequence (3.2). Since $\mathrm{Ext}^1(\Omega^\mathcal{E}, \Gamma) = 0$ also the second claim follows. \square

Lemma 3.23. *Let $\Gamma, K \in \mathrm{L}_R(\mathcal{C}, R)$ and $u: \oplus_q \Omega^{\mathcal{E}_q} \rightarrow K$ be a map, where $\mathcal{E}_q \in \mathcal{C}$. If for all $\Omega^\mathcal{E} \rightarrow K$ with $\mathcal{E} \in \mathcal{C}$ there exists a surjective map $v: \oplus_t \mathcal{E}_t \rightarrow \mathcal{E}$ with $\mathcal{E}_t \in \mathcal{C}$ such that the composition $\oplus_t \Omega^{\mathcal{E}_t} \rightarrow \Omega^\mathcal{E} \rightarrow K$ factors through u and Γ is cohomologically left exact on v then the map*

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(K, \Gamma) \rightarrow \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C}, R)}(\oplus_q \Omega^{\mathcal{E}_q}, \Gamma) \simeq \prod_q \Gamma_{\mathcal{E}_q}$$

is injective.

Proof. Let $\mathcal{E} \in \mathcal{C}$ and $x \in K_{\mathcal{E}}$, which corresponds to a map $\Omega^{\mathcal{E}} \rightarrow K$. Consider the data given by hypothesis with respect to this last map. We have commutative diagrams

$$\begin{array}{ccc} \bigoplus_t \Omega^{\mathcal{E}_t} & \xrightarrow{\Omega^v} & \Omega^{\mathcal{E}} \\ \downarrow & & \downarrow x \\ \bigoplus_q \Omega^{\mathcal{E}_q} & \xrightarrow{u} & K \end{array} \quad \begin{array}{ccc} \mathrm{Hom}(K, \Gamma) & \xrightarrow{\gamma} & \Gamma_{\mathcal{E}} \\ \downarrow \lambda & & \downarrow \delta \\ \prod_q \Gamma_{\mathcal{E}_q} & \longrightarrow & \prod_t \Gamma_{\mathcal{E}_t} \end{array}$$

where the second diagram is obtained by applying $\mathrm{Hom}(-, \Gamma)$ to the first one. The map λ is the map in the statement, while γ is the evaluation in $x \in K_{\mathcal{E}}$. Thanks to 3.22 and since Γ is cohomologically left exact on v the map δ is injective. So if $\phi \in \mathrm{Hom}(K, \Gamma)$ is such that $\lambda(\phi) = 0$ it follows that $\gamma(\phi) = \phi_{\mathcal{E}}(x) = 0$, as required. \square

Theorem 3.24. *If $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then $\mathrm{Lex}_R(\mathcal{C}, A)$ coincides with the subcategory of $\mathrm{L}_R(\mathcal{C}, A)$ of cohomologically left exact functors.*

Proof. Let $\mu = \bigoplus_j \mu_j: \bigoplus_{j \in J} \mathcal{E}_j \rightarrow \mathcal{E}$ be a surjective map with $\mathcal{E}, \mathcal{E}_j \in \mathcal{C}$ and set $\Delta = \mathrm{Im}(\Omega^{\mu})$, $K = \mathrm{Ker}(\Omega^{\mu})$. By 3.2 there exists a test sequence $\bigoplus_k \mathcal{E}_k \rightarrow \bigoplus_j \mathcal{E}_j \xrightarrow{\mu} \mathcal{E} \rightarrow 0$. Using 3.22 we have to prove that if $\Gamma \in \mathrm{L}_R(\mathcal{C}, R)$ is cohomologically left exact then $\lambda: \mathrm{Hom}(K, \Gamma) \rightarrow \prod_k \Gamma_{\mathcal{E}_k}$ is injective. We are going to apply 3.23 with respect to the map $\bigoplus_k \Omega^{\mathcal{E}_k} \rightarrow K$. If $\overline{\mathcal{E}} \in \mathcal{C}$, a map $\Omega^{\overline{\mathcal{E}}} \rightarrow K$ is a locally finite map $\overline{\mathcal{E}} \rightarrow \bigoplus_j \mathcal{E}_j$ which is zero composed by μ , or, equivalently, mapping in the image of $\bigoplus_k \mathcal{E}_k \rightarrow \bigoplus_j \mathcal{E}_j$. Consider the kernel \mathcal{H} of the difference of the maps $\bigoplus_k \mathcal{E}_k \rightarrow \bigoplus_j \mathcal{E}_j$ and $\overline{\mathcal{E}} \rightarrow \bigoplus_j \mathcal{E}_j$. Since this difference map is locally finite, $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ and using 3.2 there is a surjective map $\bigoplus_t \mathcal{E}_t \rightarrow \mathcal{H}$ with $\mathcal{E}_t \in \mathcal{C}$ such that $\bigoplus_t \mathcal{E}_t \rightarrow \bigoplus_k \mathcal{E}_k$ is locally finite and $\bigoplus_t \mathcal{E}_t \rightarrow \overline{\mathcal{E}}$ is surjective. This gives the desired factorization for applying 3.23. \square

We now show how to reduce the number of test sequences in order to check when a $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ belongs to $\mathrm{Lex}_R(\mathcal{C}, A)$. The following is the key lemma:

Lemma 3.25. *Let $\Phi' \subseteq \Phi_{\mathcal{C}}$ such that, for all $\mathcal{E} \in \mathcal{C}$ and $\Delta \in \Phi_{\mathcal{C}}(\mathcal{E})$ there exists $\Delta' \in \Phi' \cap \Phi_{\mathcal{C}}(\mathcal{E})$ such that $\Delta' \subseteq \Delta$ (inside $\Omega^{\mathcal{E}}$). If $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ and Γ is cohomologically left exact on all the elements of Φ' then Γ is cohomologically left exact.*

Proof. Consider $\Delta \in \Phi_{\mathcal{C}}(\mathcal{E})$, $\Delta' \subseteq \Delta$ with $\Delta' \in \Phi'$ and the exact sequence $0 \rightarrow \Delta/\Delta' \rightarrow \Omega^{\mathcal{E}}/\Delta' \rightarrow \Omega^{\mathcal{E}}/\Delta \rightarrow 0$. Applying $\mathrm{Hom}(-, \Gamma)$ and using that $\mathrm{Ext}^1(\Omega^{\mathcal{E}}, \Gamma) = 0$, the only non trivial vanishing to check is $\mathrm{Hom}(\Delta/\Delta', \Gamma) = 0$. Write $\Delta' = \mathrm{Im}(\Omega^u)$, where $u: \bigoplus_q \mathcal{E}_q \rightarrow \mathcal{E}$. We can apply 3.23 to $K = \Delta$ and the map $\Omega^u: \bigoplus_q \Omega^{\mathcal{E}_q} \rightarrow \Delta$: if $\Omega^{\mathcal{E}'} \rightarrow \Delta \subseteq \Omega^{\mathcal{E}}$ is a map corresponding to $\psi: \mathcal{E}' \rightarrow \mathcal{E}$, then $\psi^{-1}(\Delta') \in \Phi_{\mathcal{C}}$ and, by hypothesis, we can find $\Phi' \ni \mathrm{Im}(\Omega^v) \subseteq \psi^{-1}(\Delta')$; the last inclusion tells us that v is the factorization required for 3.23. Thus the map $\mathrm{Hom}(\Omega^u, \Gamma): \mathrm{Hom}(\Delta, \Gamma) \rightarrow \mathrm{Hom}(\bigoplus_q \Omega^{\mathcal{E}_q}, \Gamma)$ is injective and, since $\Delta' = \mathrm{Im}(\Omega^u) \subseteq \Delta$, this is also true for the restriction $\mathrm{Hom}(\Delta, \Gamma) \rightarrow \mathrm{Hom}(\Delta', \Gamma)$, whose kernel is $\mathrm{Hom}(\Delta/\Delta', \Gamma)$, as required. \square

Proposition 3.26. *Let $\mathcal{D} \subseteq \mathrm{QCoh} \mathcal{X}$ be a subcategory and $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$. If $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$ then Γ is exact on finite test sequences in \mathcal{D} and transforms arbitrary direct sums in \mathcal{D} into products. The converse holds if one of the following conditions is satisfied:*

- the category \mathcal{D} is stable by arbitrary direct sums;
- all the sheaves in \mathcal{D} are finitely presented, $\mathcal{D} \subseteq \mathrm{QCoh}^{\mathcal{D}} \mathcal{X}$ and \mathcal{X} is quasi-compact. In this case $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$ if and only if it is cohomologically left exact on all surjective maps $\mathcal{E}' \rightarrow \mathcal{E}$ with $\mathcal{E} \in \mathcal{D}$ and $\mathcal{E}' \in \mathcal{D}^{\oplus}$.

In any of the above cases, if moreover \mathcal{D} is additive and all surjections in \mathcal{D} have kernel in \mathcal{D} then $\text{Lex}_R(\mathcal{D}, A)$ is the subcategory of $\text{L}_R(\mathcal{D}, A)$ of left exact functors which transforms arbitrary direct sums in products.

Proof. If $\Gamma \in \text{Lex}_R(\mathcal{D}, A)$ then it is clearly exact on finite test sequences. Given a set $\{\mathcal{E}_j \in \mathcal{D}\}_{j \in J}$ set $\mathcal{E} = \bigoplus_j \mathcal{E}_j$. If $\mathcal{E} \in \mathcal{D}$, then the sequence

$$0 \longrightarrow \bigoplus_{j \in J} \mathcal{E}_j \xrightarrow{\text{id}_{\mathcal{E}}} \mathcal{E} \longrightarrow 0$$

is a test sequence and therefore we get that the natural map $\Gamma_{\mathcal{E}} \longrightarrow \prod_j \Gamma_{\mathcal{E}_j}$ is an isomorphism. If \mathcal{D} is stable by arbitrary direct sums is easy to see that the converse holds. Moreover the last part of the statement follows easily from the first part.

So we focus on the second point and we assume that all the sheaves in \mathcal{D} are finitely presented, $\mathcal{D} \subseteq \text{QCoh}^{\mathcal{D}} \mathcal{X}$ and that \mathcal{X} is quasi-compact. Since the class of finitely presented quasi-coherent sheaves on \mathcal{X} modulo isomorphism is a set, we can assume $\mathcal{D} = \mathcal{C}$ small. Let $\Phi' \subseteq \Phi_{\mathcal{C}}$ be the subset of functors of the form $\text{Im}(\Omega^{\mu})$ for some surjective map $\mu: \mathcal{E}' \longrightarrow \mathcal{E}$ with $\mathcal{E} \in \mathcal{C}$ and $\mathcal{E}' \in \mathcal{C}^{\oplus}$. The set Φ' satisfies the hypothesis of 3.25: if $v: \bigoplus_{j \in J} \mathcal{E}_j \longrightarrow \mathcal{E}$ is a surjective map then there exists a finite subset $J_0 \subseteq J$ such that $v|_{\mathcal{E}'}: \mathcal{E}' = \bigoplus_{j \in J_0} \mathcal{E}_j \longrightarrow \mathcal{E}$ is surjective because \mathcal{E} is of finite type and \mathcal{X} is quasi-compact. In particular, taking into account 3.24, the last claim of the second point follows. It remains to show that if $\Gamma \in \text{L}_R(\mathcal{C}, A)$ is exact on finite test sequences then Γ is cohomologically left exact on all the elements of Φ' . Let $\mu: \mathcal{E}' \longrightarrow \mathcal{E}$ be a surjective map with $\mathcal{E} \in \mathcal{C}$ and $\mathcal{E}' \in \mathcal{C}^{\oplus}$. Since \mathcal{E} is finitely presented and \mathcal{E}' is of finite type it follows that $\text{Ker}(\mu)$ is of finite type and, since \mathcal{X} is quasi-compact and $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$, there exists $\mathcal{E}'' \in \mathcal{C}^{\oplus}$ and a surjective map $\mathcal{E}'' \longrightarrow \text{Ker}(\mu)$. Thus $\mathcal{E}'' \longrightarrow \mathcal{E}' \xrightarrow{\mu} \mathcal{E} \longrightarrow 0$ is a finite test sequence and by 3.22 it follows that Γ is cohomologically left exact on μ as required. \square

There is another characterization of $\text{Lex}_R(\mathcal{C}, A)$ in terms of sheaves on a site. Although we will not use it in this paper, I think it is worth to point out. We refer to [SP014, Tag 00YW] for general definitions and properties. We start by comparing $\text{Lex}_R(\mathcal{C}, A)$ and $\text{Lex}_R(\mathcal{C}^{\oplus}, A)$.

Proposition 3.27. *If $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$ then the equivalence $\text{L}_R(\mathcal{C}^{\oplus}, A) \simeq \text{L}_R(\mathcal{C}, A)$ maps $\text{Lex}_R(\mathcal{C}^{\oplus}, A)$ to $\text{Lex}_R(\mathcal{C}, A)$.*

Proof. We can assume $A = R$. Let $\Gamma \in \text{L}_R(\mathcal{C}^{\oplus}, R)$ such that $\Gamma \in \text{Lex}_R(\mathcal{C}, R)$ and consider $\Phi' \subseteq \Phi_{\mathcal{C}^{\oplus}}$ the set of subfunctors $\Delta \subseteq \Omega^{\mathcal{E}}$ that can be written as follows: $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ and there are surjective maps $\mu_k: \bigoplus_q \mathcal{E}_{q,k} \longrightarrow \mathcal{E}_k$ for $\mathcal{E}_k, \mathcal{E}_{q,k} \in \mathcal{C}$ such that $\Delta = \text{Im}(\Omega^{\mu_1}) \oplus \cdots \oplus \text{Im}(\Omega^{\mu_r})$. Since for such Δ we have

$$\Omega^{\mathcal{E}}/\Delta \simeq \bigoplus_i (\Omega^{\mathcal{E}_i}/\Delta_i)$$

it follows that Γ is cohomologically left exact on all the elements of Φ' . Taking into account 3.24, in order to conclude that $\Gamma \in \text{Lex}_R(\mathcal{C}, R)$ we can show that $\Phi' \subseteq \Phi_{\mathcal{C}^{\oplus}}$ satisfies the hypothesis of 3.25. So let $\Delta = \text{Im}(\Omega^{\mu}) \in \Phi_{\mathcal{C}^{\oplus}}$ where $\mu: \bigoplus_q \mathcal{E}_q \longrightarrow \mathcal{E}$ where $\mathcal{E}, \mathcal{E}_q \in \mathcal{C}^{\oplus}$. If $\mathcal{E} = \mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_r$ with $\mathcal{E}'_i \in \mathcal{C}$ and $\psi_i: \mathcal{E}'_i \longrightarrow \mathcal{E}$ are the inclusions then $\Delta_i = \psi_i^{-1}(\Delta) \in \Phi_{\mathcal{C}^{\oplus}}(\mathcal{E}'_i) = \Phi_{\mathcal{C}}(\mathcal{E}'_i)$ and it is easy to see that $\Phi' \ni \Delta_1 \oplus \cdots \oplus \Delta_r \subseteq \Delta$ as required. \square

Proposition 3.28. *If $\mathcal{C} \subseteq \text{QCoh}^{\mathcal{C}} \mathcal{X}$ and \mathcal{J} is the smallest Grothendieck topology on \mathcal{C}^{\oplus} containing $\Phi_{\mathcal{C}^{\oplus}}$ then $\text{Lex}_R(\mathcal{C}^{\oplus}, A)$ is the category of sheaves of A -modules on $(\mathcal{C}^{\oplus}, \mathcal{J})$ which are R -linear.*

Proof. We can assume $A = R$ and $\mathcal{C} = \mathcal{C}^{\oplus}$. If $\Delta \subseteq \Omega^{\mathcal{E}}$ is a sieve and $f: \mathcal{E}' \longrightarrow \mathcal{E}$ we set $f^{-1}(\Delta) = \Delta \times_{\Omega^{\mathcal{E}}} \Omega^{\mathcal{E}'} \subseteq \Omega^{\mathcal{E}'}$. Let $\tilde{\mathcal{J}}$ be the set of sieves $\Delta \subseteq \Omega^{\mathcal{E}}$ of \mathcal{C} such that, for all

$\Gamma \in \text{Lex}_R(\mathcal{C}, R)$ and maps $f: \mathcal{E}' \rightarrow \mathcal{E}$ the map

$$\text{Hom}_{(\text{Sets})}(\Omega^{\mathcal{E}'}, \Gamma) \rightarrow \text{Hom}_{(\text{Sets})}(f^{-1}(\Delta), \Gamma)$$

is bijective. Here $\text{Hom}_{(\text{Sets})}$ denotes the set of natural transformation of functors with values in (Sets) . The set $\tilde{\mathcal{J}}$ is a Grothendieck topology on \mathcal{C} such that all functors in $\text{Lex}_R(\mathcal{C}, R)$ are sheaves. Notice that, by 2.15, if $A, B \in \text{L}_R(\mathcal{C}, R)$ then $\text{Hom}_{(\text{Sets})}(A, B) = \text{Hom}_{\text{L}_R(\mathcal{C}, R)}(A, B)$. Moreover if $\Delta \in \Phi_{\mathcal{C}}$ and $f: \mathcal{E}' \rightarrow \mathcal{E}$ is a map in \mathcal{C} then $\Delta' = f^{-1}(\Delta) \in \Phi_{\mathcal{C}}$ and if $\Gamma \in \text{L}_R(\mathcal{C}, R)$ then, applying $\text{Hom}_{\text{L}_R(\mathcal{C}, R)}(-, \Gamma)$ on the exact sequence $0 \rightarrow \Delta' \rightarrow \Omega^{\mathcal{E}'} \rightarrow \Omega^{\mathcal{E}'} / \Delta' \rightarrow 0$ and taking into account that $\text{Ext}^1(\Omega^{\mathcal{E}'}, \Gamma) = 0$ we obtain an exact sequence

$$0 \rightarrow \text{Hom}(\Omega^{\mathcal{E}'} / \Delta', \Gamma) \rightarrow \text{Hom}_{(\text{Sets})}(\Omega^{\mathcal{E}'}, \Gamma) \rightarrow \text{Hom}_{(\text{Sets})}(\Delta', \Gamma) \rightarrow \text{Ext}^1(\Omega^{\mathcal{E}'} / \Delta', \Gamma) \rightarrow 0$$

Thus, if \mathcal{J} is the smallest topology on \mathcal{C} containing $\Phi_{\mathcal{C}}$ then $\Phi_{\mathcal{C}} \subseteq \mathcal{J} \subseteq \tilde{\mathcal{J}}$ and everything follows easily from 3.24. \square

We now apply 3.18 and 3.26 in some (more) concrete situations.

Theorem 3.29. *The category $\text{Lex}_R(\text{QCoh } \mathcal{X}, A)$ is the category of contravariant, R -linear and left exact functors $\Gamma: \text{QCoh } \mathcal{X} \rightarrow \text{Mod } A$ which transform arbitrary direct sums in products. Moreover the functor*

$$\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{Lex}_R(\text{QCoh } \mathcal{X}, A)$$

is an equivalence of categories.

Proof. Follows from 3.18 and 3.26 with $\mathcal{D} = \text{QCoh } \mathcal{X}$. \square

Theorem 3.30. *Let \mathcal{X} be a noetherian algebraic stack. The category $\text{Lex}_R(\text{Coh } \mathcal{X}, A)$ is the category of contravariant, R -linear and left exact functors $\text{Coh } \mathcal{X} \rightarrow \text{Mod } A$. Moreover the functor*

$$\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{Lex}_R(\text{Coh } \mathcal{X}, A)$$

is an equivalence of categories.

Proof. Follows from 3.18 and 3.26, taking into account that in our assumptions $\text{Coh } \mathcal{X}$ is an abelian category that generates $\text{QCoh } \mathcal{X}$. \square

Theorem 3.31. *Assume that \mathcal{X} is quasi-compact and that $\text{Loc } \mathcal{X}$ generates $\text{QCoh } \mathcal{X}$. Then $\text{Lex}_R(\text{Loc } \mathcal{X}, A)$ is the category of contravariant, R -linear and left exact functors $\text{Loc } \mathcal{X} \rightarrow \text{Mod } A$. Moreover the functor*

$$\Omega^*: \text{QCoh}_A \mathcal{X} \rightarrow \text{Lex}_R(\text{Loc } \mathcal{X}, A)$$

is an equivalence of categories.

Proof. Follows from 3.18 and 3.26, taking into account that all surjections in $\text{Loc } \mathcal{X}$ have kernels in $\text{Loc } \mathcal{X}$. \square

Theorem 3.32. *Let B be an R -algebra and $\mathcal{D} \subseteq \text{Mod } B$ be a subcategory that generates $\text{Mod } B$, that is there exists $\mathcal{E}_1, \dots, \mathcal{E}_r \in \mathcal{D}$ with a surjective map $\bigoplus_i \mathcal{E}_i \rightarrow B$. Then the functor*

$$\Omega^*: \text{Mod}(A \otimes_R B) \rightarrow \text{Lex}_R(\mathcal{D}, A)$$

is an equivalence of categories. Moreover if $\mathcal{D} \subseteq \text{Loc } B$ then $\text{Lex}_R(\mathcal{D}, A) = \text{L}_R(\mathcal{D}, A)$.

Proof. If $\mathcal{X} = \text{Spec } B$, then $\text{QCoh}_A \mathcal{X} \simeq \text{Mod}(A \otimes_R B)$ and the first part follows from 3.18. For the last claim, observe that any $\Gamma: \text{Loc } B \rightarrow \text{Mod } A$ is exact because any short exact sequence in $\text{Loc } B$ splits. By 3.31 we can conclude that $\text{L}_R(\text{Loc } B, A) = \text{Lex}_R(\text{Loc } B, A)$. If now $\mathcal{D} \subseteq \text{Loc } B$ and $\Gamma \in \text{L}_R(\mathcal{D}, A)$, we can extend it to $\bar{\Gamma} \in \text{L}_R(\text{Loc } B, A)$ and therefore $\Gamma = \bar{\Gamma}|_{\mathcal{D}} \in \text{Lex}_R(\mathcal{D}, A)$. \square

We want to extend Theorem 3.18 to functors with monoidal structures.

Definition 3.33. If \mathcal{D} is a monoidal subcategory of $\mathrm{QCoh} \mathcal{X}$ we define $\mathrm{PMLex}_R(\mathcal{D}, A)$ (resp. $\mathrm{MLex}_R(\mathcal{D}, A)$) as the subcategory of $\mathrm{PML}_R(\mathcal{D}, A)$ (resp. $\mathrm{ML}_R(\mathcal{D}, A)$) of functors Γ such that $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$.

Theorem 3.34. Let \mathcal{D} be a monoidal subcategory of $\mathrm{QCoh} \mathcal{X}$ that generates it. Then the functors

$$\Omega^*: \mathrm{Rings}_A \mathcal{X} \longrightarrow \mathrm{PMLex}_R(\mathcal{D}, A) \text{ and } \Omega^*: \mathrm{QAlg}_A \mathcal{X} \longrightarrow \mathrm{MLex}_R(\mathcal{D}, A)$$

(see 2.22) are equivalence of categories. If \mathcal{D} is small a quasi inverse is given by $\mathcal{A}_{*, \mathcal{D}}: \mathrm{PMLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{Rings}_A \mathcal{X}$ and $\mathcal{A}_{*, \mathcal{D}}: \mathrm{MLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{QAlg}_A \mathcal{X}$ respectively (see 2.23). Moreover if $\overline{\mathcal{D}} \subseteq \mathcal{D}$ is a monoidal subcategory that generates $\mathrm{QCoh} \mathcal{X}$ the restriction functors $\mathrm{PMLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{PMLex}_R(\overline{\mathcal{D}}, A)$ and $\mathrm{MLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{MLex}_R(\overline{\mathcal{D}}, A)$ are equivalences.

Proof. Assume that \mathcal{D} is small. Then $\Omega^*: \mathrm{Rings}_A \mathcal{X} \longrightarrow \mathrm{PMLex}_R(\mathcal{D}, A)$ and $\mathcal{A}_{*, \mathcal{D}}: \mathrm{PMLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{Rings}_A \mathcal{X}$ are quasi-inverses of each other because, by 2.23, we have natural transformations $\mathrm{id} \xrightarrow{\gamma} \Omega^* \circ \mathcal{A}_{*, \mathcal{D}}$ and $\mathcal{A}_{*, \mathcal{D}} \circ \Omega^* \xrightarrow{\beta} \mathrm{id}$ which are isomorphisms thanks to 3.18. In particular if $\overline{\mathcal{D}} \subseteq \mathcal{D}$ is a monoidal subcategory that generates $\mathrm{QCoh} \mathcal{X}$ then the restriction functor $\mathrm{PMLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{PMLex}_R(\overline{\mathcal{D}}, A)$ is an equivalence. Since γ and β preserve unities by 2.23, the same holds if we replace $\mathrm{Rings}_A \mathcal{X}$ by $\mathrm{QAlg}_A \mathcal{X}$ and $\mathrm{PMLex}_R(\mathcal{D}, A)$ by $\mathrm{MLex}_R(\mathcal{D}, A)$. Notice that there exists a small subcategory $\mathcal{C}' \subseteq \mathcal{D}$ that generates $\mathrm{QCoh} \mathcal{X}$ thanks to 3.17. If $I \subseteq \mathcal{D}$ is a set we set \mathcal{C}_I for the category containing all tensor products with factors in $\mathcal{C}' \cup I$ and $\mathcal{O}_{\mathcal{X}}$. We have that \mathcal{C}_I is a collection of small monoidal subcategories of \mathcal{D} , that generate $\mathrm{QCoh} \mathcal{X}$ and such that $\mathcal{C}_I \subseteq \mathcal{C}_{I'}$ if $I \subseteq I'$. If $\mathcal{C} = \mathcal{C}_{\emptyset}$ we can show that the restrictions $\mathrm{PMLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{PMLex}_R(\mathcal{C}, A)$ and $\mathrm{MLex}_R(\mathcal{D}, A) \longrightarrow \mathrm{MLex}_R(\mathcal{C}, A)$ are equivalences by proceeding as in the proof of 3.18. All the other claims in the statement follow easily from this fact. \square

Theorem 3.35. The theorems 3.29, 3.30, 3.31 continue to hold if we replace Lex_R by PMLex_R (resp. MLex_R), $\mathrm{QCoh}_A \mathcal{X}$ by $\mathrm{QRings}_A \mathcal{X}$ (resp. $\mathrm{QAlg}_A \mathcal{X}$) and the word “functors” by “pseudo-monoidal functors” (resp. “monoidal functors”).

4. GROUP SCHEMES AND REPRESENTATIONS.

Let G be a flat and affine group scheme over R . In this section we want to interpret the results obtained in the case $\mathcal{X} = \mathrm{B}_R G$, the stack of G -torsors for the fpqc topology, which is a quasi-compact fpqc stack with affine diagonal.

If A is an R -algebra, by standard theory we have that $\mathrm{QCoh}_A \mathrm{B}_R G$ is the category $\mathrm{Mod}^G A$ of G -comodules over A . Recall that the regular representation $R[G]$ of G is by definition the G -comodule $p_* \mathcal{O}_G$. By definition it comes equipped with a morphism of R -algebras $\varepsilon: R[G] \longrightarrow R$ induced by the unit section of G .

Remark 4.1. If $M \in \mathrm{Mod}^G A$ then the composition

$$(M \otimes_R R[G])^G \longrightarrow M \otimes_R R[G] \xrightarrow{\mathrm{id}_M \otimes \varepsilon} M$$

is an isomorphism. This follows from [Jan87, 3.4] applied to $G = H$.

We start with a criterion to find a set of generators for $\mathrm{QCoh} \mathrm{B}_R G$.

Proposition 4.2. If the regular representation $R[G]$ is a filtered direct limit of modules $B_i \in \mathrm{Mod}^G R$ which are finitely presented as R -modules then $\{B_i^\vee\}_{i \in I}$ generates $\mathrm{Mod}^G R$.

Proof. Set $B = R[G]$ and $\varepsilon_i: B_i \longrightarrow R$ for the composition $B_i \longrightarrow B \xrightarrow{\varepsilon} R$ and let $M \in \mathrm{Mod}^G R$. Since filtered direct limits commute with tensor products and taking invariants, by 4.1 we have that the limit of the maps $(\varepsilon_i \otimes \mathrm{id}_M)_{(B_i \otimes M)^G}: (B_i \otimes M)^G \longrightarrow M$ is an isomorphism.

This means that for any $m \in M$ there exists $i_m \in I$ and an element $\psi_m \in (B_{i_m} \otimes M)^G$ such that $(\varepsilon_{i_m} \otimes \text{id}_M)(\psi_m) = m$. The map $B_{i_m} \otimes M \rightarrow \text{Hom}(B_{i_m}^\vee, M)$ is G -equivariant and therefore we obtain a $\delta_m \in \text{Hom}^G(B_{i_m}^\vee, M)$ such that $\delta_m(\varepsilon_{i_m}) = m$. This implies that the map

$$\bigoplus_{m \in M} \delta_m : \bigoplus_{m \in M} B_{i_m}^\vee \rightarrow M$$

is surjective and therefore that M is generated by $\{B_{i_m}^\vee\}_{i_m \in I}$. \square

Remark 4.3. The class \mathcal{G}_R of flat, affine group schemes G over R such that $R[G]$ is a direct limit of modules in $\text{Loc}(B_R G)$ is stable by arbitrary products and projective limits. Moreover by construction contains all groups which are flat, finite and finitely presented over R , i.e. $R[G] \in \text{Loc}(B_R G)$, and thus all profinite groups. Since any G -comodule is the union of the sub G -comodules which are finitely generated R -modules (see [Jan87, 2.13]), we see that \mathcal{G}_R contains all flat groups defined over a Dedekind domain or a field, such as GL_r , SL_r and all diagonalizable groups. Proposition 4.2 tells us that if $G \in \mathcal{G}_R$ then $B_R G$ has the resolution property.

Let A be an R -algebra. We denote by $\text{Loc}^G A$ the subcategory of $\text{Mod}^G A$ of G -comodules that are locally free of finite rank (projective of finite type) as A -modules, so that $\text{Loc}(B_R G) \simeq \text{Loc}^G R$. We define $\text{QAdd}^G A$ ($\text{QPMon}^G A$, $\text{QMon}^G A$) as the category of *covariant* R -linear (pseudo-monoidal, monoidal) functors $\text{Loc}^G R \rightarrow \text{Mod} A$. We set $\text{QRings}^G A$ for the category of $M \in \text{Mod}^G A$ with a G -equivariant map $M \otimes_A M \rightarrow M$ and $\text{QAlg}^G A$ for the (not full) subcategory of $\text{QRings}^G A$ of commutative R -algebras.

Definition 4.4. The group G is called *linearly reductive* if the functor $(-)^G : \text{Mod}^G R \rightarrow \text{Mod} R$ is exact.

Remark 4.5. If G is linearly reductive then any short exact sequence in $\text{Loc}^G R$ splits. Indeed if $M \rightarrow N$ is surjective then $\text{Hom}_R^G(N, M) \rightarrow \text{Hom}_R^G(N, N)$ is surjective, yielding a G -equivariant section $N \rightarrow M$.

Theorem 4.6. *If $B_R G$ has the resolution property then the functors*

$$\text{Mod}^G A \rightarrow \text{QAdd}^G A, \text{QRings}^G A \rightarrow \text{QPMon}^G A, \text{QAlg}^G A \rightarrow \text{QMon}^G A$$

which maps M to the functor $(- \otimes_R M)^G : \text{Loc}^G R \rightarrow \text{Mod} A$ are well defined, fully faithful and have essential image the subcategory of functors which are left exact on short exact sequences in $\text{Loc}^G R$. In particular they are equivalences if G is a linearly reductive group.

Proof. Set $\mathcal{C} = \text{Loc}^G R$. The functor $(-)^G : \text{Loc}^G R \rightarrow \text{Loc}^G R$ is an equivalence and therefore we get equivalences $\text{QAdd}^G A \simeq \text{L}_R(\mathcal{C}, A)$, $\text{QPMon}^G A \simeq \text{PML}_R(\mathcal{C}, A)$ and $\text{QMon}^G A \simeq \text{ML}_R(\mathcal{C}, A)$. Left exact functors are sent to left exact functors. Under those equivalences Ω^M corresponds to $(- \otimes_R M)^G$ because $\text{Hom}_{B_R G}(\mathcal{E}^\vee, M) \simeq H^0(\mathcal{E} \otimes_{\mathcal{O}_{B_R G}} M) \simeq (\mathcal{E} \otimes_R M)^G$. Thus the result follows from 3.12, 3.31, 3.35 and 4.5. \square

5. TANNAKA RECONSTRUCTION FOR STACKS WITH THE RESOLUTION PROPERTY.

Definition 5.1. A (contravariant) monoidal functor $\Omega : \mathcal{C} \rightarrow \mathcal{D}$ between symmetric monoidal categories is called *strong* if the maps

$$\Omega_V \otimes \Omega_W \rightarrow \Omega_{V \otimes W}$$

are isomorphisms for all $V, W \in \mathcal{C}$ and the map $J \rightarrow F_I$ is an isomorphism, where I and J are the unities of \mathcal{C} and \mathcal{D} respectively.

In this section we want to understand what sheaves of algebras correspond to strong monoidal functors in the equivalence of 3.34, in the case where \mathcal{D} is a subcategory of locally free sheaves.

We will consider only fpqc stacks with quasi-affine diagonal, for instance algebraic stacks with quasi-affine diagonal (see [MBL99, Corollary 10.7]) and quasi-separated schemes. This is because resolution property is somehow meaningless for other stacks, see for instance Remark (1) in the introduction of [Tot04].

Definition 5.2. If \mathcal{X} is a fiber category over R , $\mathcal{C} \subseteq \text{Loc } \mathcal{X}$ is a monoidal subcategory and A is an R -algebra we define $\text{SMex}_R(\mathcal{C}, A)$ as the subcategory of $\text{MLex}_R(\mathcal{C}, A)$ of functors Γ which are strong monoidal and, for all geometric points $\text{Spec } k \rightarrow \text{Spec } A$, $\Gamma \otimes_A k \in \text{MLex}_R(\mathcal{C}, k)$. Given a fiber category \mathcal{Y} we denote by $\text{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y})$ the category of *covariant*, R -linear and strong monoidal functors $\Gamma: \mathcal{C} \rightarrow \text{Loc } \mathcal{Y}$ which are exact on all exact sequences $\mathcal{E}'' \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow 0$ with $\mathcal{E} \in \mathcal{C}$ and $\mathcal{E}', \mathcal{E}'' \in \mathcal{C}^\oplus$. We also define $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ as the fiber category over R whose fiber over an R -algebra A is $\text{Fib}_{\mathcal{X}, \mathcal{C}}(\text{Spec } A)$ and we call $\mathcal{P}_{\mathcal{C}}$ the functor

$$(5.1) \quad \mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}, (\text{Spec } A \xrightarrow{s} \mathcal{X}) \mapsto (s^*: \mathcal{C} \rightarrow \text{Loc } A)$$

We will prove the following:

Theorem 5.3. *Let \mathcal{X} be a quasi-compact fpqc stack over R with quasi-affine diagonal, A be an R -algebra and $\mathcal{C} \subseteq \text{Loc } \mathcal{X}$ be a monoidal subcategory with duals that generates $\text{QCoh } \mathcal{X}$. Then the functors*

$$\begin{aligned} (\text{Spec } \mathcal{A}_{\Gamma, \mathcal{C}} \rightarrow \mathcal{X}) &\longleftarrow | (\Gamma: \mathcal{C} \rightarrow \text{Mod } A) \\ \mathcal{X}(A) &\longrightarrow \text{SMex}_R(\mathcal{C}, A) \\ (s: \text{Spec } A \rightarrow \mathcal{X}) &\longmapsto ((s^*_{|\mathcal{C}})^\vee: \mathcal{C} \rightarrow \text{Loc } A) \end{aligned}$$

are well defined and quasi-inverses of each other. In particular the functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$ is an equivalence of stacks.

An immediate corollary and generalization of Theorem 5.3 is the following.

Corollary 5.4. *Let \mathcal{X} be a quasi-compact fpqc stack over R with quasi-affine diagonal, $\mathcal{C} \subseteq \text{Loc } \mathcal{X}$ be a monoidal subcategory with duals that generates $\text{QCoh } \mathcal{X}$ and \mathcal{Y} be a fibered category over R . Then the functor*

$$\text{Hom}(\mathcal{Y}, \mathcal{X}) \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}(\mathcal{Y}), (\mathcal{Y} \xrightarrow{f} \mathcal{X}) \mapsto f^*_{|\mathcal{C}}: \mathcal{C} \rightarrow \text{Loc } (\mathcal{Y})$$

is an equivalence of categories.

Proof. The map in the statement is obtained applying $\text{Hom}(\mathcal{Y}, -)$ to the functor $\mathcal{P}_{\mathcal{C}}: \mathcal{X} \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}$, which is an equivalence by 5.3. \square

One of the key points in the proof of statements above is a characterization of the following stacks.

Definition 5.5. A *pseudo-affine* stack is a quasi-compact fpqc stack with quasi-affine diagonal such that all quasi-coherent sheaves on it are generated by global sections. A map $f: \mathcal{Y}' \rightarrow \mathcal{Y}$ of fibered categories is called *pseudo-affine* if for all maps $T \rightarrow \mathcal{Y}$ from an affine scheme the fiber product $T \times_{\mathcal{Y}} \mathcal{Y}'$ is pseudo-affine.

Theorem 5.6. *Intersection of quasi-compact open subschemes (thought of as sheaves) of affine schemes are pseudo-affine. Conversely if U is a pseudo-affine stack then it is (equivalent to) a sheaf and it is the intersection of the quasi-compact open subschemes of $\text{Spec } H^0(\mathcal{O}_U)$ containing it.*

A quasi-affine scheme is pseudo-affine and, conversely, a pseudo-affine stack which is algebraic is quasi-affine (see [Gro13, Proposition 3.1]). In general a pseudo-affine sheaf is not quasi-affine. An example is the sheaf intersection of all the complement of closed points in $\mathrm{Spec} k[x, y]$, where k is a field.

The last statement of Theorem 5.3 admits an almost converse. Let \mathcal{X} be a fibered category over R and $\mathcal{C} \subseteq \mathrm{Loc} \mathcal{X}$ be a full monoidal subcategory. We have a functor $\mathcal{G}_*: \mathcal{C} \longrightarrow \mathrm{Loc}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$ which maps a $\mathcal{E} \in \mathcal{C}$ to the locally free sheaf

$$\mathcal{G}_{\mathcal{E}}: \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^{\mathrm{op}} \longrightarrow (\mathrm{Ab}), \quad \mathcal{G}_{\mathcal{E}}(\Gamma \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(A)) = \Gamma_{\mathcal{E}} \text{ for all } R\text{-algebras } A$$

In particular $\mathcal{P}_{\mathcal{C}}^* \mathcal{G}_{\mathcal{E}} \simeq \mathcal{E}$ for $\mathcal{E} \in \mathcal{C}$. Notice that, a priori, $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is not necessarily fibered in groupoids and therefore the notion of a locally free sheaf on it is not defined (although one can easily guess the definition). Part of Theorems 5.3 and 5.7 (below) is to prove that, under suitable conditions on \mathcal{C} , this is indeed true.

Theorem 5.7. *Let \mathcal{X} be a quasi-compact fibered category over R and $\mathcal{C} \subseteq \mathrm{Loc} \mathcal{X}$ be a full monoidal subcategory with duals and with $\mathrm{Sym}^n \mathcal{E} \in \mathcal{C}$ for all $n \in \mathbb{N}$ if \mathcal{E} has local rank not invertible in R . If $f: \mathcal{C} \longrightarrow \mathbb{N}$ is a function then $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$, the sub-fibered category of $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ of functors Γ such that $\mathrm{rk} \Gamma_{\mathcal{E}} = f(\mathcal{E})$ for all $\mathcal{E} \in \mathcal{C}$, is a quotient stack of a pseudo-affine sheaf by the action of a (possibly infinite) products of GL_n ; in particular it is a quasi-compact fpqc stack in groupoids with affine diagonal. Moreover the subcategory $\{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}} \subseteq \mathrm{Loc}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f)$ generates $\mathrm{QCoh}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f)$ and, in particular, $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$ has the resolution property.*

Let \mathcal{I} be the set of functions $f: \mathcal{C} \longrightarrow \mathbb{N}$ such that there exists a geometric point $s: \mathrm{Spec} L \longrightarrow \mathcal{X}$ with $f(\mathcal{E}) = \mathrm{rk} s^* \mathcal{E}$. Given $f: \mathcal{C} \longrightarrow \mathbb{N}$ then $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f \neq \emptyset$ if and only if $f \in \mathcal{I}$ and, if \mathcal{I} is finite, then

$$\mathrm{Fib}_{\mathcal{X}, \mathcal{C}} = \bigsqcup_{f \in \mathcal{I}} \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$$

In particular if \mathcal{I} is finite then $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is a quasi-compact fpqc stack in groupoids with affine diagonal, the subcategory $\{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}} \subseteq \mathrm{Loc}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$ generates $\mathrm{QCoh}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$ and $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ has the resolution property.

Remark 5.8. The condition that \mathcal{I} is finite in 5.7 is not optimal, but at least it covers the case where \mathcal{X} admits a surjective (on equivalence classes of geometric points) map $\mathcal{X}' \longrightarrow \mathcal{X}$ from an algebraic stack whose connected components are open (e.g. \mathcal{X} is a connected or Noetherian algebraic stack). It is not clear if $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ always has the resolution property.

We start with a first characterization of pseudo-affine stacks. Notice that the condition that all quasi-coherent sheaves on \mathcal{X} are generated by global section exactly means that the category $\{\mathcal{O}_{\mathcal{X}}\}$ generates $\mathrm{QCoh} \mathcal{X}$.

Proposition 5.9. *Let $\mathcal{X} \xrightarrow{\pi} \mathrm{Spec} R$ be a quasi-compact fpqc stack with quasi-affine diagonal. Then the following conditions are equivalent:*

- 1) *the stack \mathcal{X} is pseudo-affine;*
- 2) *the map $\pi^* \pi_* \mathcal{F} \longrightarrow \mathcal{F}$ is surjective for all $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$;*
- 3) *the stack \mathcal{X} is equivalent to a sheaf and there exists a flat monomorphism $\mathcal{X} \longrightarrow \mathrm{Spec} B$, where B is a ring.*

In this case the map $p: \mathcal{X} \longrightarrow \mathrm{Spec} H^0(\mathcal{O}_{\mathcal{X}})$ is a flat monomorphism, $p_: \mathrm{QCoh} \mathcal{X} \longrightarrow \mathrm{Mod} H^0(\mathcal{X})$ is fully faithful and $p^* p_* \simeq \mathrm{id}$. Moreover if $\mathcal{X} \times_{H^0(\mathcal{O}_{\mathcal{X}})} k \neq \emptyset$ for all geometric points $\mathrm{Spec} k \longrightarrow \mathrm{Spec} H^0(\mathcal{O}_{\mathcal{X}})$ then p is an isomorphism.*

Proof. 2) \implies 1). Given $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$, take a surjective map $R^{(I)} \rightarrow \pi_* \mathcal{F}$. In this case the composition

$$\mathcal{O}_{\mathcal{X}}^{(I)} \simeq \pi^* R^{(I)} \rightarrow \pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$$

is surjective.

1) \implies 2). A sheaf $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$ is generated by global sections and the image of $\pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$ contains all of them.

3) \implies 1). Denote by $p: \mathcal{X} \rightarrow \mathrm{Spec} B$ the flat monomorphism. We are going to show that $\delta_{\mathcal{F}}: p^* p_* \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism for all $\mathcal{F} \in \mathrm{QCoh} \mathcal{X}$. Arguing as in 2) \implies 1) this will conclude the proof. By hypothesis there exists a representable fpqc covering $h: \mathrm{Spec} C \rightarrow \mathcal{X}$ and we must prove that $h^* \delta_{\mathcal{F}}$ is an isomorphism. Let $f = ph: \mathrm{Spec} C \rightarrow \mathrm{Spec} B$ be the composition, which is flat by hypothesis, and consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec} C & \xrightarrow{s} & \mathcal{X} \times_B C & \xrightarrow{\alpha} & \mathrm{Spec} C \\ & \searrow h & \downarrow & & \downarrow f \\ & & \mathcal{X} & \xrightarrow{p} & \mathrm{Spec} B \end{array}$$

Since α is a monomorphism with a section, α and s are inverses of each other. Using the fact that f is flat the map $h^* \delta_{\mathcal{F}}$ is given by

$$p^* p_* \mathcal{F}(\mathrm{Spec} C \xrightarrow{h} \mathcal{X}) = p_* \mathcal{F}(\mathrm{Spec} C \xrightarrow{f} \mathrm{Spec} B) = \mathcal{F}(\mathcal{X} \times_B C \rightarrow \mathcal{X}) \xrightarrow{s^*} \mathcal{F}(\mathrm{Spec} C \xrightarrow{h} \mathcal{X})$$

and therefore it is an isomorphism.

1) \implies 3). Set $B = H^0(\mathcal{O}_{\mathcal{X}})$ and $p: \mathcal{X} \rightarrow \mathrm{Spec} B$ the induced map. Notice that $L_B(\{\mathcal{O}_{\mathcal{X}}\}, B) \simeq \mathrm{Mod} B$ and under this isomorphism $\Omega^*: \mathrm{QCoh} \mathcal{X} \rightarrow L_B(\{\mathcal{O}_{\mathcal{X}}\}, B)$ and $\mathcal{F}_{*, \{\mathcal{O}_{\mathcal{X}}\}}: L_B(\{\mathcal{O}_{\mathcal{X}}\}, B) \rightarrow \mathrm{QCoh} \mathcal{X}$ correspond to $p_*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{Mod} B$ and $p^*: \mathrm{Mod} B \rightarrow \mathrm{QCoh} \mathcal{X}$ respectively. By hypothesis, 3.12 and 3.18 the map $p: \mathcal{X} \rightarrow \mathrm{Spec} B$ is flat, $p_*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{Mod} B$ is fully faithful and $p^* p_* \simeq \mathrm{id}$.

We want to show that $\mathcal{X} \rightarrow \mathrm{Spec} B$ is fully faithful or, equivalently, that the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times_B \mathcal{X}$ is an equivalence. Let $V = \mathrm{Spec} C \rightarrow \mathcal{X}$ be a representable fpqc covering and denote by $s: V \rightarrow \mathcal{X} \times_B V$ the graph of h . We have Cartesian diagrams

$$\begin{array}{ccc} V \times_{\mathcal{X}} V & \xrightarrow{q} & V \times_B V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_B \mathcal{X} \end{array} \quad \begin{array}{ccc} V \times_{\mathcal{X}} V & \xrightarrow{q} & V \times_B V \\ \downarrow & & \downarrow \\ V & \xrightarrow{s} & \mathcal{X} \times_B V \end{array}$$

and the vertical arrows are representable fpqc coverings. By descent it follows that Δ is an equivalence if we prove that s is an equivalence.

Let $f: V \rightarrow \mathcal{X} \xrightarrow{p} \mathrm{Spec} B$ be the composition. Since p is flat also f is flat and, in particular, $H^0(\mathcal{O}_{\mathcal{X} \times_B V}) = C$. Since $\mathcal{X} \times_B V \rightarrow \mathcal{X}$ is affine it follows that $\{\mathcal{O}_{\mathcal{X} \times_B V}\}$ generates $\mathrm{QCoh}(\mathcal{X} \times_B V)$. Since s is a section of $\mathcal{X} \times_B V \rightarrow V$, we see that we can assume that $p: \mathcal{X} \rightarrow \mathrm{Spec} B$ has a section, that we still denote by $s: \mathrm{Spec} B \rightarrow \mathcal{X}$. In this case we have to prove that p or s is an equivalence. Notice that this implies the last claim in the statement. Indeed if $\mathcal{X} \times_B k \neq \emptyset$ for all geometric points then $f: V \rightarrow \mathrm{Spec} B$ is an fpqc covering: it follows that the map $\mathcal{X} \rightarrow \mathrm{Spec} B$ is an equivalence because it has this property fpqc locally.

We first prove that $p_*: \mathrm{QCoh} \mathcal{X} \rightarrow \mathrm{Mod} B$ is an equivalence. It suffices to show that, if $M \in \mathrm{Mod} B$, then the map $\gamma_M: M \rightarrow p_* p^* M$ is an isomorphism. Notice that $p^* \gamma_M$ is a section of the map $(p^* p_*) p^* M \rightarrow p^* M$ which is an isomorphism. So $p^* \gamma_M$ and $\gamma_M = s^* p^* \gamma_M$ are isomorphisms. Since $p_* s_*: \mathrm{Mod} B \rightarrow \mathrm{Mod} B$ is the identity, we can conclude that $s_* \simeq p^*$ and

$s^* \simeq p_*$. Consider the Cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & \mathrm{Spec} C \\ \downarrow t & & \downarrow h \\ \mathrm{Spec} B & \xrightarrow{s} & \mathcal{X} \end{array}$$

where h is a representable fpqc covering. Since \mathcal{X} has quasi-affine diagonal it follows that U is a quasi-affine scheme. Moreover

$$H^0(\mathcal{O}_U) \simeq g_* t^* \mathcal{O}_B \simeq h^* s_* \mathcal{O}_B \simeq h^* p^* \mathcal{O}_B \simeq C$$

Thus $g: U \rightarrow \mathrm{Spec} C$ and $s: \mathrm{Spec} B \rightarrow \mathcal{X}$ are open immersions. We prove that g is a surjective, which imply that g and s are isomorphisms. Let Z be the complement of U in $\mathrm{Spec} C$ with reduced structure. We have

$$H^0(\mathcal{O}_Z) \simeq p_* h_* \mathcal{O}_Z \simeq s^* h_* \mathcal{O}_Z = t_* g^* \mathcal{O}_Z = 0$$

Thus Z is empty as required. \square

Remark 5.10. The assumption on the diagonal in 5.9 is necessary: the stack $\mathcal{X} = B_k E$, where E is an elliptic curve over a field k , is not a sheaf but $\mathrm{QCoh} \mathcal{X} \simeq \mathrm{QCoh} k$.

Remark 5.11. If $f: \mathcal{Y} \rightarrow \mathcal{X}$ is a pseudo-affine map of pseudo-algebraic fiber categories then it is quasi-compact with affine diagonal and the map $f^* f_* \mathcal{F} \rightarrow \mathcal{F}$ is surjective for all $\mathcal{F} \in \mathrm{QCoh} \mathcal{Y}$. In particular if $\mathcal{D} \subseteq \mathrm{QCoh} \mathcal{X}$ generates $\mathrm{QCoh} \mathcal{X}$ then $f^* \mathcal{D}$ generates $\mathrm{QCoh} \mathcal{Y}$. The first claims follow by standard argument of descent while the second by considering an atlas of \mathcal{X} and using the characterization 3) of 5.9.

If $\mathcal{Y} \rightarrow \mathrm{Spec} A$ is a map of fibered categories then \mathcal{Y} is pseudo-affine if and only if $\mathcal{Y} \rightarrow \mathrm{Spec} A$ is pseudo-affine, because if \mathcal{Y} is pseudo-affine and $g: \mathcal{Y}' \rightarrow \mathcal{Y}$ is an affine map, then \mathcal{Y}' is an fpqc sheaf with quasi-affine diagonal and $\mathcal{O}_{\mathcal{Y}'} = g^* \mathcal{O}_{\mathcal{Y}}$ generates $\mathrm{QCoh} \mathcal{Y}'$.

Finally, if $f: \mathcal{Y}' \rightarrow \mathcal{Y}$ is a map of fpqc stacks and $Y \rightarrow \mathcal{Y}$ is an fpqc atlas such that $Y' = Y \times_{\mathcal{Y}} \mathcal{Y}' \rightarrow Y$ is pseudo-affine then f is pseudo-affine. Indeed by standard arguments of descent we can assume $\mathcal{Y} = \mathrm{Spec} B$ and $Y = \mathrm{Spec} B'$ affine, which also implies that \mathcal{Y}' is quasi-compact with affine diagonal. Moreover since $B \rightarrow B'$ is flat we have $H^0(\mathcal{O}_{Y'}) \simeq H^0(\mathcal{O}_{\mathcal{Y}'} \otimes_B B')$ and therefore we can assume that $H^0(\mathcal{O}_{\mathcal{Y}'}) = B$ and $H^0(\mathcal{O}_{Y'}) = B'$. In this case $Y' \rightarrow Y$ is flat and fully faithful and, since \mathcal{Y} is an fpqc stack, it follows that also $\mathcal{Y}' \rightarrow \mathcal{Y}$ is a flat and fully faithful.

Remark 5.12. Let G be a flat and affine group scheme over R such that $B_R G$ has the resolution property. Taking into account 5.11, if U is a pseudo-affine sheaf over R with an action of G then $[U/G]$ has the resolution property because the map $[U/G] \rightarrow B_R G$ is pseudo-affine. The same conclusion follows for a stack $\mathcal{X} = [X/G]$, where X is a scheme, if there exists $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ whose pullback \mathcal{M} to X is very ample relatively to R . Indeed \mathcal{X} can be written as $[U/G \times \mathbb{G}_m]$ where U is the complement of the zero section of $\mathcal{M} \rightarrow X$: the fact that \mathcal{M} is very ample tells us that U is quasi-affine. Moreover $B(G \times \mathbb{G}_m)$ has the resolution property. Indeed let N be the canonical invertible sheaf on $B_R \mathbb{G}_m$ and $\mathcal{F} \in \mathrm{QCoh}(B_R G)$. The action of \mathbb{G}_m yields a decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$$

and all $\mathcal{F}_n = (\mathcal{F} \otimes N^{\otimes -n})^{\mathbb{G}_m}$ are sub $G \times \mathbb{G}_m$ -representations. If $x \in \mathcal{F}_n$ there exists $\mathcal{E} \in \mathrm{Loc}(B_R G)$ and a G -equivariant map $\mathcal{E} \rightarrow \mathcal{F}_n$ with x in its image. Thus the map $\mathcal{E} \otimes N^{\otimes n} \rightarrow \mathcal{F}_n$, where \mathcal{E} has the trivial action of \mathbb{G}_m , is $G \times \mathbb{G}_m$ -equivariant and has x in its image.

The following property is known for algebraic stacks (see [Gro13, Corollary 5.11] and [Tot04, Proposition 1.3]).

Corollary 5.13. *A quasi-compact fpqc stack \mathcal{X} with quasi-affine diagonal and with the resolution property has affine diagonal.*

Proof. Since \mathcal{X} is pseudo-algebraic the category $\text{Loc } \mathcal{X}$ is essentially small. Thus we can consider a set \mathcal{R} of representatives of locally free sheaves over \mathcal{X} . Given $\mathcal{E} \in \mathcal{R}$ we define the sheaf

$$\text{Fr}(\mathcal{E}): (\text{Sch}/\mathcal{X})^{\text{op}} \longrightarrow (\text{Sets}), \quad \text{Fr}(\mathcal{E})(U \xrightarrow{f} \mathcal{X}) = \bigsqcup_{n \in \mathbb{N}} \text{Iso}_U(\mathcal{O}_U^n, f^* \mathcal{E})$$

It is easy to see that $\text{Fr}(\mathcal{E}) \longrightarrow \mathcal{X}$ is an affine fpqc covering. In particular $g: \text{Fr} = \prod_{\mathcal{E} \in \mathcal{R}} \text{Fr}(\mathcal{E}) \longrightarrow \mathcal{X}$ is also an affine fpqc covering and Fr is quasi-compact with quasi-affine diagonal. Thus it is enough to show that Fr has affine diagonal. Since g is affine by 5.11 $g^* \text{Loc } \mathcal{X}$ generates $\text{QCoh } \text{Fr}$. On the other hand if $\mathcal{E} \in \text{Loc } \mathcal{X}$ then by construction Fr is a (finite) disjoint union of open substacks over which $g^* \mathcal{E}$ is free, which implies that $g^* \mathcal{E}$ is generated by global sections. We can conclude that Fr is a pseudo-affine sheaf and thus has affine diagonal. \square

Proof. (of Theorem 5.3). Since \mathcal{X} has affine diagonal by 5.13, all morphisms $\text{Spec } A \longrightarrow \mathcal{X}$ are affine. Therefore the functor $\mathcal{X}(A) \longrightarrow \text{QAlg}_A \mathcal{X}$ which maps $s: \text{Spec } A \longrightarrow \mathcal{X}$ to $s_* \mathcal{O}_A$ is fully faithful. By 3.34 and the fact that

$$\Omega_{\mathcal{E}}^{s_* \mathcal{O}_A} = \text{Hom}(\mathcal{E}, s_* \mathcal{O}_A) \simeq (s^* \mathcal{E})^\vee \text{ for } s \in \mathcal{X}(A), \mathcal{E} \in \mathcal{C}$$

we can conclude that the functor $\mathcal{X}(A) \longrightarrow \text{SMex}_R(\mathcal{C}, A)$, $s \mapsto (s^* \mathcal{E})^\vee$ is well defined and fully faithful. Thus everything follows if we prove that, given $\Gamma \in \text{SMex}_R(\mathcal{C}, A)$, the composition $p: \text{Spec } \mathcal{A}_{\Gamma, \mathcal{C}} \longrightarrow \mathcal{X}_A \longrightarrow \text{Spec } A$ is an isomorphism. Set $\mathcal{A} = \mathcal{A}_{\Gamma, \mathcal{C}}$, $\mathcal{Y} = \text{Spec } \mathcal{A}$ and $f: \mathcal{Y} \longrightarrow \mathcal{X}$ for the structure morphism. We want to apply 5.9 on $p: \mathcal{Y} \longrightarrow \text{Spec } A$. Notice that \mathcal{Y} is quasi-compact and has affine diagonal. Moreover by 3.34

$$\Gamma_{\mathcal{E}} \simeq \Omega_{\mathcal{E}}^{\mathcal{A}} = \text{Hom}(\mathcal{E}, \mathcal{A}) = \text{Hom}(\mathcal{E}, f_* \mathcal{O}_{\mathcal{Y}}) \simeq \text{Hom}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{Y}}) = H^0((f^* \mathcal{E})^\vee)$$

In particular, since $\Gamma_{\mathcal{O}_{\mathcal{X}}} \simeq A$, the map $A \longrightarrow H^0(\mathcal{A})$ is an isomorphism. We show now that, if $\text{Spec } k \longrightarrow \text{Spec } A$ is a geometric point, then $\mathcal{Y} \times_A k \neq \emptyset$. If $g: \mathcal{X}_k \longrightarrow \mathcal{X}_A$ is the projection then $\mathcal{Y} \times_A k \simeq \text{Spec}(g^* \mathcal{A})$, while by 2.12 we have $g^* \mathcal{A} \simeq \mathcal{A}_{\Gamma \otimes_A k, \mathcal{C}}$. By 3.34 we get $\Gamma \otimes_A k \simeq \Omega_{\Gamma \otimes_A k, \mathcal{C}}$ and therefore $\mathcal{A}_{\Gamma \otimes_A k, \mathcal{C}} = 0$, that is $\mathcal{Y} \times_A k = \emptyset$, implies $\Gamma \otimes_A k = 0$, while $\Gamma_{\mathcal{O}_{\mathcal{X}}} \otimes_A k = A \otimes_A k \simeq k$.

It remains to show that $\{\mathcal{O}_{\mathcal{Y}}\}$ generates $\text{QCoh } \mathcal{Y}$. Since $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ is affine, $f^* \mathcal{C}$ generates $\text{QCoh } \mathcal{Y}$ by 5.11. Thus we have to prove that all the sheaves $f^* \mathcal{E}$ are generated by global sections. By hypothesis the map

$$H^0(f^* \mathcal{E}'^\vee) \otimes H^0(f^* \mathcal{E}^\vee) \longrightarrow H^0(f^*(\mathcal{E}' \otimes \mathcal{E})^\vee)$$

is an isomorphism for all $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$. Since \mathcal{C} has duals, choosing $\mathcal{E}' = \mathcal{E}^\vee$ the above map became the evaluation

$$\omega: H^0(f^* \mathcal{E}) \otimes \text{Hom}_{\mathcal{Y}}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{Y}}) \longrightarrow \text{End}_{\mathcal{Y}}(f^* \mathcal{E})$$

Since ω is an isomorphism there exist $x_1, \dots, x_n \in H^0(f^* \mathcal{E})$, $\phi_1, \dots, \phi_n \in \text{Hom}(f^* \mathcal{E}, \mathcal{O}_{\mathcal{Y}})$ such that $\text{id}_{f^* \mathcal{E}} = \omega(\sum_i x_i \otimes \phi_i)$. This implies that the map $\mathcal{O}_{\mathcal{Y}}^n \longrightarrow f^* \mathcal{E}$ given by the global sections x_1, \dots, x_n is surjective, as required.

For the last statement we claim that $-\vee: \text{Fib}_{\mathcal{X}, \mathcal{C}}(A) \longrightarrow \text{SMex}_R(\mathcal{C}, A)$, $\Gamma \longmapsto \Gamma^\vee$ is an equivalence. Taking into account 3.26, if $\Gamma \in \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$ then $\Gamma^\vee \in \text{SMex}_R(\mathcal{C}, A)$ because Γ is exact on all finite test sequences and the dual of a right exact sequence of locally free sheaves is again exact. Moreover the map $\mathcal{X}(A) \longrightarrow \text{SMex}_R(\mathcal{C}, A)$ factors as

$$\mathcal{X}(A) \xrightarrow{\mathcal{P}\mathcal{C}} \text{Fib}_{\mathcal{X}, \mathcal{C}}(A) \xrightarrow{-\vee} \text{SMex}_R(\mathcal{C}, A)$$

Thus $-^\vee: \text{Fib}_{\mathcal{X}, \mathcal{C}}(A) \rightarrow \text{SMex}_R(\mathcal{C}, A)$ is essentially surjective and $-^\vee: \text{SMex}_R(\mathcal{C}, A) \rightarrow \text{Fib}_{\mathcal{X}, \mathcal{C}}(A)$ is a quasi-inverse. \square

Proof. (of Theorem 5.6) Let $X = \text{Spec } B$ be an affine scheme, $\{U_i\}_{i \in I}$ be a set of quasi-compact open subsets of X and set $U = \cap_i U_i$. If $i \in I$ the subscheme U_i is the complement of the zero locus of finitely many element of B and thus there exists a free B -module E_i and a map $\phi_i: E_i \rightarrow B$ such that U_i is the locus where ϕ_i is surjective. Let $V: \text{Aff}/B \rightarrow (\text{Sets})$ be the functor

$$V(A) = \{(s_i)_{i \in I} \mid s_i \in E_i \otimes_B A \mid \phi_i(s_i) = 1\}$$

which is an affine scheme. The map $V \rightarrow \text{Spec } B$ factors through U and $V \rightarrow U$ is surjective (as functors). Moreover if $\text{Spec } A \rightarrow U$ is a map then $V \times_U A = V \times_X A$ because $U \rightarrow X$ is a monomorphism. Since $V \times_X A$ is isomorphic to $(\prod_i \text{Ker } \phi_i) \times_X A$ we can conclude that $V \rightarrow U$ is an affine fpqc epimorphism, so that U is quasi-compact, and that $U \rightarrow X$ is flat. The result then follows from 5.9.

For the converse, denote by Z the intersection in the statement, set $\mathcal{C} = \{\mathcal{O}_U\}$, $B = H^0(\mathcal{O}_U)$ and let $\alpha: \text{Spec } A \rightarrow \text{Spec } B$ be a map which factors through Z and $\mathcal{T}_*: \mathcal{O}_U^n \rightarrow \mathcal{O}_U^m \rightarrow \mathcal{O}_U \rightarrow 0$ be an exact sequence on U . Since $\mathcal{C} = \{\mathcal{O}_{\text{Spec } B}\}$ as monoidal categories, by 5.3 it is enough to show that $\alpha^*: \mathcal{C} \rightarrow \text{Loc } A$ is exact on \mathcal{T}_* . The sequence \mathcal{T}_* defines a complex \mathcal{W}_* of free A -modules, namely $\mathcal{W}_* = H^0(\mathcal{T}_*)$, and the locus W in $\text{Spec } A$ where \mathcal{W}_* is exact is quasi-compact, open and contains U . Thus $Z \subseteq W$, the sequence \mathcal{W}_* become exact on Z and therefore α^* maintains its exactness, as required. \square

Lemma 5.14. *Let \mathcal{R} be a set, $f: \mathcal{R} \rightarrow \mathbb{N}$ be a map and set*

$$\text{GL}_f = \prod_{i \in \mathcal{R}} \text{GL}_{f(i)}$$

and \mathcal{F}_i for the locally free sheaf of rank $f(i)$ on B GL_f coming from the universal one on $\text{B GL}_{f(i)}$. Then the subcategory of $\text{Loc}(\text{B GL}_f)$ consisting of all tensor products of sheaves

$$\{\mathcal{F}_i\}_{i \in \mathcal{R}}, \{(\text{Sym}^m \mathcal{F}_i)^\vee\}_{i \in \mathcal{R} \text{ s.t. } f(i) \notin R^*, m \in \mathbb{N}}$$

generates $\text{QCoh}(\text{B GL}_f)$.

Proof. We are going to apply 4.2. Let \mathcal{D} be the subcategory of Loc B GL_f generated by direct sums and tensor products of the sheaves $\text{Sym}^m \mathcal{F}_i, (\det \mathcal{F}_i)^{-1}$ for $i \in \mathcal{R}, m \in \mathbb{N}$. We claim that $R[\text{GL}_f]$ is a direct limit of representations in \mathcal{D} . The representation $R[\text{GL}_f]$ is a direct limit of tensor products of the regular representations $R[\text{GL}_{f(i)}]$ for $i \in \mathcal{R}$. This allows us to reduce to the case of GL_n (i.e. \mathcal{R} has one element). Call \mathcal{F} the universal rank n sheaf on B GL_n and \mathcal{G} the free R -module of rank n . As usual we can write $R[\text{GL}_n] = R[X_{uv}]_{\det}$ for $1 \leq u, v \leq n$, where \det is the determinant polynomial. The R -submodule generated by the $X_{u,v}$ is a GL_n -subrepresentation isomorphic to $\mathcal{F} \otimes \mathcal{G}$. In particular we obtain injective maps

$$\text{Sym}^{nm}(\mathcal{F} \otimes \mathcal{G}) \otimes (\det \mathcal{F})^{-\otimes m} \rightarrow R[\text{GL}_n]$$

whose image is the set of fractions f / \det^m where f is homogeneous of degree nm . Those images form an increasing sequence of sub representations saturating $R[\text{GL}_n]$. Thus $R[\text{GL}_n]$ is the direct limit of the sheaves $\text{Sym}^{nm}(\mathcal{F} \otimes \mathcal{G}) \otimes (\det \mathcal{F})^{-\otimes m}$ which belongs to \mathcal{D} .

Coming back to the general framework, by 4.2 and the existence of surjective maps $\mathcal{F}_i^{\otimes f(i)} \rightarrow \det \mathcal{F}_i$ and, if $f(i) \in R^*$ so that $\text{GL}_{f(i)}$ is linearly reductive, $(\mathcal{F}_i^\vee)^{\otimes m} \rightarrow (\text{Sym}^m \mathcal{F}_i)^\vee$ we get the desired result. \square

Proof. (of Theorem 5.7). It is easy to see that $\text{Fib}_{\mathcal{X}, \mathcal{C}}$ is a stack (not necessarily in groupoids) for the fpqc topology on Aff/R . To avoid problems with disjoint unions we can assume that \mathcal{X}

is a Zariski stack. We will use notation from 5.14. Let \mathcal{R} be a set of representatives of \mathcal{C}/\simeq with $\mathcal{O}_{\mathcal{X}} \in \mathcal{R}$, J be a finite subset of \mathcal{R} and denote by \mathcal{I}_J the set of $f \in \mathbb{N}^J$ extending to a function of \mathcal{I} . Given $f \in \mathbb{N}^J$ we denote by \mathcal{X}_f the open locus of \mathcal{X} where $\mathrm{rk} \mathcal{E} = f(\mathcal{E})$ for all $\mathcal{E} \in J$. Notice that $\mathcal{X} = \bigsqcup_{f \in \mathcal{I}_J} \mathcal{X}_f$ and, since \mathcal{X} is quasi-compact, \mathcal{I}_J is finite. The sheaves $(\mathcal{E}|_{\mathcal{X}_f})_{\mathcal{E} \in J}$ induces a map $\mathcal{X}_f \rightarrow \mathrm{BGL}_f = \mathcal{B}_f$ and thus a map

$$\omega_J: \mathcal{X} \rightarrow \bigsqcup_{f \in \mathcal{I}_J} \mathcal{B}_f = \mathcal{B}_J$$

For all $\mathcal{E} \in J$ there is a locally free sheaf $\mathcal{H}_{\mathcal{E},J}$ on \mathcal{B}_J such that $(\mathcal{H}_{\mathcal{E},J})|_{\mathcal{B}_f}$ is the canonical locally free sheaf of rank $f(\mathcal{E})$ (pullback from $\mathrm{BGL}_{f(\mathcal{E})}$), so that $\omega_J^* \mathcal{H}_{\mathcal{E},J} \simeq \mathcal{E}$. Let \mathcal{D}_J be the subcategory of $\mathrm{Loc}(\mathcal{B}_J)$ consisting of all tensor products and duals of $\mathcal{H}_{\mathcal{E},J}$ and, if there exists $f \in \mathcal{I}_J$ such that $f(\mathcal{E}) \notin R^*$, also of $\mathrm{Sym}^n \mathcal{H}_{\mathcal{E},J}$ for $n \in \mathbb{N}$ and $\mathcal{E} \in J$. We have $\omega_J^* \mathcal{D}_J \subseteq \mathcal{C}$ by construction, that \mathcal{D}_J generates $\mathrm{QCoh}(\mathcal{B}_J)$ by 5.14 and, in particular, that $\mathcal{P}_{\mathcal{D}_J}: \mathcal{B}_J \rightarrow \mathrm{Fib}_{\mathcal{B}_J, \mathcal{D}_J}$ is an equivalence of stacks by 5.3. When $J = \{\mathcal{E}\}$ we will replace J by \mathcal{E} in the subscripts.

We now show that $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is a stack in groupoids. If $\Gamma, \Gamma' \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(A)$, $\delta: \Gamma \rightarrow \Gamma'$ is a morphism and $\mathcal{E} \in \mathcal{C}$ then $\delta_{\mathcal{E}}: \Gamma_{\mathcal{E}} \rightarrow \Gamma'_{\mathcal{E}}$ is an isomorphism because $\Gamma \circ \omega_{\mathcal{E}}^* \xrightarrow{\delta \circ \omega_{\mathcal{E}}^*} \Gamma' \circ \omega_{\mathcal{E}}^*$ is a morphism in the groupoid $\mathrm{Fib}_{\mathcal{B}_{\mathcal{E}}, \mathcal{D}_{\mathcal{E}}}(A) \simeq \mathcal{B}_{\mathcal{E}}(A)$.

Given $f: \mathcal{C} \rightarrow \mathbb{N}$, we show that $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f \neq \emptyset$ if and only if $f \in \mathcal{I}$. For the if part, if $f \in \mathcal{I}$ there exists $s: \mathrm{Spec} L \rightarrow \mathcal{X}$ such that $\mathrm{rk} s^* \mathcal{E} = f(\mathcal{E})$ for all $\mathcal{E} \in \mathcal{C}$. Thus $s|_{\mathcal{C}} \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$. For the only if part we show that, if L is an algebraically closed field and $\Gamma \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(L)$ then $\mathrm{rk} \Gamma_*: \mathcal{C} \rightarrow \mathbb{N}$ belongs to \mathcal{I} . This will also show that, if \mathcal{I} is finite, then $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ is the disjoint union of the $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$ for $f \in \mathcal{I}$: indeed $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$ is the locus where $\mathrm{rk} \mathcal{G}_{\mathcal{E}} = f(\mathcal{E})$ for all $\mathcal{E} \in \mathcal{C}$.

So let $\Gamma \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}(L)$. Given $J \subseteq \mathcal{R}$ finite consider $\Gamma \circ \omega_J^* \in \mathrm{Fib}_{\mathcal{B}_J, \mathcal{D}_J}(L) \simeq \mathcal{B}_J(L)$, so that there exists a map $\mathrm{Spec} L \xrightarrow{s} \mathcal{B}_J$ such that $\Gamma \circ \omega_J^* \simeq s^*|_{\mathcal{D}_J}$. The map s has image in some component \mathcal{B}_f with $f \in \mathcal{I}_J$. In particular if $\mathcal{E} \in J$ we have

$$\mathrm{rk} \Gamma_{\mathcal{E}} = \mathrm{rk} \Gamma \omega_J^* \mathcal{H}_{\mathcal{E},J} = \mathrm{rk}(s^*(\mathcal{H}_{\mathcal{E},J})|_{\mathcal{B}_f}) = f(\mathcal{E})$$

This shows that for all finite subsets $J \subseteq \mathcal{R}$ we have that $f_J = (\mathrm{rk} \Gamma_*)|_J$ belongs to \mathcal{I}_J . By construction \mathcal{X}_{f_J} is a non-empty (open and) closed substack of \mathcal{X} . Since $\mathcal{X}_{f_J} \subseteq \mathcal{X}_{f_{J'}}$ if $J' \subseteq J$ and \mathcal{X} is quasi-compact it follows that $\bigcap_J \mathcal{X}_{f_J} \neq \emptyset$ and thus that $\mathrm{rk} \Gamma_* \in \mathcal{I}$.

It remains to show the claims about $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$ when $f \in \mathcal{I}$. The sheaf $\mathcal{G}_{\mathcal{E}}$ has rank $f(\mathcal{E})$ on $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f$ and thus we get a map $g: \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f \rightarrow \mathrm{BGL}_f$ such that $g^* \mathcal{F}_{\mathcal{E}} \simeq \mathcal{G}_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{R}$ (here we are using notation from 5.14). Let \mathcal{C}' be the subcategory of $\mathrm{Loc} \mathrm{BGL}_f$ obtained by taking tensor products and duals of the sheaves $\mathcal{F}_{\mathcal{E}}$ and, if $f(\mathcal{E}) \notin R^*$, also of $\mathrm{Sym}^n \mathcal{F}_{\mathcal{E}}$ for $n \in \mathbb{N}$ and $\mathcal{E} \in \mathcal{R}$. Moreover set $\mathcal{D} = \{\mathcal{G}_{\mathcal{E}}\}_{\mathcal{E} \in \mathcal{C}} \subseteq \mathrm{Loc}(\mathrm{Fib}_{\mathcal{X}, \mathcal{C}})$. We claim that $g^* \mathcal{C}' \subseteq \mathcal{D}|_{\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f}$. It suffices to show that if $\mathcal{E} \in \mathcal{C}$ then $\mathcal{G}_{\mathcal{E}^\vee} \simeq (\mathcal{G}_{\mathcal{E}})^\vee$ and, if $f(\mathcal{E}) \notin R^*$, $\mathcal{G}_{\mathrm{Sym}^n \mathcal{E}} \simeq \mathrm{Sym}^n \mathcal{G}_{\mathcal{E}}$ over the whole $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$. Consider $J = \{\mathcal{E}\} \subseteq \mathcal{R}$. The functor $\omega_{\mathcal{E}}: \mathcal{X} \rightarrow \mathcal{B}_{\mathcal{E}}$ induces $\omega_{\mathcal{E}}^*: \mathcal{D}_{\mathcal{E}} \rightarrow \mathcal{C}$ and a functor $\delta: \mathrm{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow \mathrm{Fib}_{\mathcal{B}_{\mathcal{E}}, \mathcal{D}_{\mathcal{E}}} \simeq \mathcal{B}_{\mathcal{E}}$. If t is either $(-)^{\vee}$ or Sym^n and $\Gamma \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}$ we have isomorphisms

$$\mathcal{G}_{t(\mathcal{E})}(\Gamma) = \Gamma_{t(\mathcal{E})} \simeq (\Gamma \circ \omega_{\mathcal{E}}^*)_{t(\mathcal{H}_{\mathcal{E}, \{\mathcal{E}\}})} \simeq \delta(\Gamma)^* t(\mathcal{H}_{\mathcal{E}, \{\mathcal{E}\}}) \simeq t(\delta(\Gamma)^* \mathcal{H}_{\mathcal{E}, \{\mathcal{E}\}}) \simeq t(\Gamma_{\mathcal{E}}) = t(\mathcal{G}_{\mathcal{E}})(\Gamma)$$

natural in Γ and thus that $\mathcal{G}_{t(\mathcal{E})} \simeq t(\mathcal{G}_{\mathcal{E}})$.

Using 5.14 and 5.11 all the claims in the statement follow if we show that the fiber product U of $g: \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f \rightarrow \mathrm{BGL}_f$ along the canonical map $\mathrm{Spec} R \rightarrow \mathrm{BGL}_f$ is pseudo-affine. Since $\mathrm{Loc}(\mathcal{X})$ is essentially small, it is easy to find a sub monoidal subcategory $\tilde{\mathcal{C}}$ of \mathcal{C} which is small and such that $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is an equivalence. Via restriction we get an equivalence $\mathrm{Fib}_{\mathcal{X}, \mathcal{C}} \rightarrow \mathrm{Fib}_{\mathcal{X}, \tilde{\mathcal{C}}}$ and we can assume that \mathcal{C} is small. If A is an R -algebra then $U(A)$ is the groupoid of $\Gamma \in \mathrm{Fib}_{\mathcal{X}, \mathcal{C}}^f(A)$ together with basis of $\Gamma_{\mathcal{E}}$ for all $\mathcal{E} \in \mathcal{C}$. In particular it is easy to see that U is (equivalent to) a

sheaf. Let $V: \text{Aff}/R \rightarrow (\text{Sets})$ be the functor which maps an R -algebra A to the set of R -linear and strong monoidal functors $\Gamma: \mathcal{C} \rightarrow \text{Loc } A$ such that $\text{rk } \Gamma_{\mathcal{E}} = f(\mathcal{E})$ together with a basis of $\Gamma_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{C}$. The sheaf V is affine because it is a closed subscheme of

$$\bar{V} = \coprod_{\mathcal{E} \rightarrow \mathcal{E}' \in \text{Arr}(\mathcal{C})} \underline{\text{Hom}}(R^{f(\mathcal{E})}, R^{f(\mathcal{E}')}) \times \coprod_{\mathcal{E}, \mathcal{E}' \in \mathcal{C}} \underline{\text{Iso}}(R^{f(\mathcal{E})} \otimes R^{f(\mathcal{E}')} , R^{f(\mathcal{E} \otimes \mathcal{E}')}) \times \mathbb{G}_{m,R}: \text{Aff}/R \rightarrow (\text{Sets})$$

which is affine. Write $V = \text{Spec } B$ and denote by $\Gamma: \mathcal{C} \rightarrow \text{Loc } B$ the canonical R -linear and strong monoidal functor. Given a finite test sequence \mathcal{T} in \mathcal{C} the sequence of maps $\Gamma_{\mathcal{T}}$ is a complex of free B -modules and denote by $V_{\mathcal{T}}$ the locus in V where this complex is exact. Clearly U is the intersection of the $V_{\mathcal{T}}$ for all finite test sequences \mathcal{T} . Since one can easily check that $V_{\mathcal{T}}$ is a quasi-compact open subscheme of V the result follows from 5.6. \square

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